

# Background Independent String Field Theory

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## Abstract

We develop a new background independent Moyal star formalism in bosonic open string field theory, rendering it into a more transparent and computationally more efficient theory. The new star product is formulated in a half-phase-space, and because phase space is independent of any background fields that the string propagates in, the interactions expressed with the new star product are background independent. The interaction written in this basis has a large amount of symmetry, including a supersymmetry  $\text{OSp}(d|2)$  that acts on matter and ghost degrees of freedom, and simplifies computations. The BRST operator that defines the quadratic kinetic term of string field theory may be regarded as the solution of the equation of motion  $\bar{A} \star \bar{A} = 0$  of a purely cubic background independent string field theory. We find an infinite number of non-perturbative solutions to this equation, and are able to associate them to the BRST operator of conformal field theories on the worldsheet. Thus, the background emerges from a spontaneous-type breaking of a purely cubic highly symmetric theory. The form of the BRST field breaks the symmetry in a tractable way such that the symmetry continues to be useful in practical perturbative computations as an expansion around some background. The new Moyal basis is called the  $\sigma$ -basis, where  $\sigma$  is the worldsheet parameter of an open string. A vital part of the new star product is a natural and crucially needed mid-point regulator in this continuous basis, so that all computations are finite. The regulator is removed after renormalization and then the theory is finite only in the critical dimension. Boundary conditions for D-branes at the endpoints of the string are naturally introduced and made part of the theory as simple rules in algebraic computations. The formalism is tested by computing some perturbative quantities and finding agreement with previous methods. We are now prepared for new non-perturbative computations. A byproduct of our approach is an astonishing suggestion of the formalism: the roots of ordinary quantum mechanics may originate in the rules of non-commutative interactions in string theory.

Keywords: String field theory, Moyal star product, BRST charge, non-perturbative string theory.

## Contents

<b>I. Introduction</b>	4
<b>II. What is the Moyal <math>\star</math> product in MSFT?</b>	8
A. The new $\star$ product	10
<b>III. Moyal <math>\star</math> in QM as inspiration for string joining</b>	13
A. String Joining iQM versus QM	17
<b>IV. Regulator</b>	24
A. Regulated delta functions	25
B. Midpoint not treated separately	28
<b>V. Representations of CFT operators in MSFT</b>	30
A. Map from CFT to MSFT and operator products	31
B. Ghosts	33
C. Stress tensor, BRST current and BRST operator	36
1. Regulated QM operators	37
2. iQM Representation of Regulated QM Operators	39
D. The MSFT action	42
E. Siegel gauge	46
F. $\text{OSp}(d 2)$ Supersymmetry Acting on Matter and Ghosts	50
G. Effective non-Perturbative Purely Cubic Quantum Action	52
<b>VI. Illustrations with Flat Space CFT</b>	55
A. Oscillators in $\sigma$ -space and Perturbative Vacuum	55
B. The Monoid in the $\sigma$ -basis	59
<b>VII. Outlook</b>	62
<b>Acknowledgments</b>	65
<b>VIII. Appendix</b>	65
A. Old Discrete Basis Versus New $\sigma$ -Basis	65
B. The BRST Gauge Transformations for an Invariant Action	71



## I. INTRODUCTION

In this paper we develop string field theory (SFT) that is independent of backgrounds. The progress is due to a new new background independent star product to describe string-string interactions. This  $\star$  product improves the mathematical structure and the computational framework of open string field theory (SFT) [1] under general conditions, including curved spacetimes or more general string backgrounds consistent with conformal symmetry on the worldsheet.

The basic approach in the current paper is similar to the Moyal star formulation of string field theory (MSFT) previously constructed in flat space-time [2]-[5]. However, now we develop the Moyal star product  $\star$  in a new basis which is independent of any string backgrounds; this is the  $\sigma$ -basis for the open string degrees of freedom  $X^M(\sigma)$  as parametrized by the worldsheet parameter  $\sigma$ , with  $0 \leq \sigma \leq \pi$ , at a fixed value of  $\tau$ . In this approach, rather than expressing the string field as a functional  $\psi(X)$  of the full string coordinate  $X^M(\sigma)$ , the string field is taken to be a functional  $A(x_+, p_-)$  of *half of the phase space* of the string, where  $x_+^M(\sigma)$  is the symmetric part of  $X^M(\sigma)$  under reflections relative to the midpoint,  $x_+^M(\sigma) = \frac{1}{2}(X^M(\sigma) + X^M(\pi - \sigma))$ , while  $p_{-M}(\sigma) = \frac{1}{2}(P_M(\sigma) - P_M(\pi - \sigma))$  is the antisymmetric part of the momentum density  $P_M = \partial S_{string} / (\partial_\tau X^M)$ , where the string action  $S_{string}$  corresponds to a conformal field theory (CFT) with any set of background fields consistent with the conformal symmetry of the worldsheet. Note that  $p_-(\sigma)$  is the canonical conjugate to  $x_-(\sigma)$  and commutes with  $x_+(\sigma)$  in the first quantization of the string. Thus the field  $A(x_+, p_-)$  is related to the field  $\psi(X) = \psi(x_+, x_-)$  by a Fourier transform from  $x_-$  to  $p_-$ . String joining in position space  $\psi(X(\sigma))$  is represented by the new Moyal product in the basis  $A(x_+(\sigma), p_-(\sigma))$ . This  $\star$  product is expressed in the half phase space without any reference to the details of the CFT, thus being independent of any backgrounds.

A second new feature is that, the BRST operator  $\hat{Q}$  for any conformal field theory is now represented as an anticommutator in MSFT,  $\hat{Q}A(x, p) = \{Q(x, p), A(x, p)\}_\star$ , purely in terms of only the new star product  $\star$ , involving the string field  $A(x, p)$  with another special string field  $Q(x, p)$  that represents the operation of  $\hat{Q}$  on  $A$ . We give an explicit expression for the *string field*  $Q(x, p)$  for any corresponding conformal field theory (CFT). This field

$Q(x, p)$  satisfies (not as an operator  $\hat{Q}^2 = 0$ , but as a string field)

$$Q(x, p) \star Q(x, p) = 0. \quad (1.1)$$

In this formulation of string field theory, computations can be carried out purely algebraically by starting from the MSFT action, without ever referring to conformal field theory, thus avoiding the complexities of conformal maps which is the difficult part of computations in other approaches to string field theory [1][8].

A third new feature is the introduction of D-brane boundary conditions at the end points of the string in the SFT context. In our formalism part of the information about D-branes can be introduced in a natural way through simple alterations in the algebraic computational procedure as we will indicate.

A major simplification in structural clarity and computational technique in MSFT is obtained in the current paper. The new formulation is not only more transparent but it also displays larger symmetries that mix the matter and the ghost degrees of freedom, such as the supersymmetry  $\text{OSp}(d|2)$ . The quadratic  $A \star A$ , cubic  $A \star A \star A$ , or any higher products of the string field all have the higher symmetry of the star product for any CFT. However, the BRST operator is less symmetric due to the structure of ghosts versus matter. For this reason, the quadratic kinetic term of string field theory breaks the higher symmetry of the cubic interactions. Nevertheless, in the Siegel gauge, in the case of flat  $d = 26$  background, there is an accidental  $\text{OSp}(26|2)$  supersymmetry which greatly simplifies computations involving ghosts.

Moreover, it is possible to rearrange the *effective quantum SFT* action (where  $\bar{A}$  includes all ghost numbers) into a purely cubic term  $S_{eff} = \frac{1}{3g_0} \int (\bar{A} \star \bar{A} \star \bar{A})$ , with  $\bar{A} = (g_0 A + Q)$ , and  $Q \star Q = 0$  as in (1.1). This form of the effective action displays the full symmetry, and is *background independent thanks to the background independence of the new star product*. Then the emergent quadratic term,  $\int (A \star Q \star A)$  in the expansion in powers of  $g_0$ , is viewed as coming from a spontaneous-type breaking that rearranges the purely cubic theory  $\bar{A}^3$  into a perturbative expansion around a background-dependent perturbative vacuum defined by the string field  $Q(x, p)$ . The purely cubic form of the action is the best way to see the background independence of MSFT which is possible because of the background independent new  $\star$  product.

Although the motivation for developing the new formalism is to deal with the issues of

strings in curved spaces (in particular cosmological backgrounds) and generalizations such as supersymmetry (which we are working on), it should not escape the reader that the formalism sheds new light and develops new tools of computation that apply also in the flat perturbative theory. Indeed, the new Moyal product in the  $\sigma$ -basis, if taken with only trivial flat-space string background, reproduces more efficiently the same computational results as the Moyal  $\star$  product in previous bases [2],[9], or the original star product that relies on the CFT formalism on the worldsheet [1].

MSFT is already known to successfully describe string theory in flat space-time by using the previous Moyal  $\star$  product [2]. For example, it yielded the 4-tachyon perturbative off-shell string scattering amplitudes (beyond the on-shell Veneziano model) in copious detail not available before the computation in [5]. Furthermore, MSFT led to the development of analytic techniques [3][4][5] for computing the nonperturbative vacuum of SFT [4]. Although the non-perturbative program remained incomplete at that time, we will indicate how similar analytic methods will work much better with the new Moyal star product. In both perturbative and non-perturbative cases analytic results in string theory that were not obtained before with other approaches were presented, thus demonstrating the usefulness of MSFT as a tool that produces new analytic results in string theory. In this paper the good features of MSFT that led to successes are preserved while MSFT is generalized in several directions thanks to the simplifications introduced by the  $\sigma$ -basis for the star product and the BRST operator. The rest of this paper is organized as follows.

- In section (II) we explain the new Moyal product that represents string joining or splitting. This is an improvement over the old discrete Moyal star product [2]. The new product operates on string fields  $A(x, p)$  which are labeled by half of the phase space of the string  $(x^M(\sigma), p_M(\sigma))$  which is denoted in low case letters  $(x, p)$ , in contrast to the full phase space  $(X^M(\sigma), P_M(\sigma))$  which is denoted in capital letters  $(X, P)$ .
- In section (III) we discuss the similarities and differences of the Moyal product [10] in ordinary quantum mechanics (QM) versus the Moyal product in string field theory which represents string joining. Although  $(\hat{x}(\sigma), \hat{p}(\sigma))$  is part of the phase space that consists of only commuting operators in QM, this same half phase space becomes non-commutative under the Moyal star  $\star$  product that represents string joining. The

non-commutativity which is due to string interactions induces a QM-like system which we call induced quantum mechanics (iQM). Based on the similarities between QM and iQM Moyal products, we identify an algebraic system in the half phase space  $(x, p)$  whose properties are just like quantum mechanics. The rules of iQM become the guiding principle for the rest of the paper for constructing the interacting string field theory as well as for performing practical computations.

- In section (IV) we discuss the regularization that is essential to resolve midpoint singularities in explicit computations in string field theory. We give the regulated and final version of the *background independent* star product, and provide an example of how the regulated  $\star$  product is used for computations. The regulator is removed after renormalization of the cubic coupling constant  $g_0$  as shown in [5]. In this section we also indicate how D-branes with non-trivial boundary conditions at the ends of the string are introduced and made part of the new formalism. In Appendix-A it is shown that if the background CFT is the flat background then the Moyal star product in the new  $\sigma$ -basis can be explicitly related to the old discrete Moyal basis [2] in which computations of interacting strings were performed in the past [3][4][5], thus showing that the new star product reproduces all of the previously successful computations.
- In section (V) we show how the elementary quantum operators of the first quantized string, including the full phase space for matter and  $b, c$  ghosts,  $(X^M(\sigma), P_M(\sigma))$ , are represented in terms of only the new string-joining Moyal product in terms of the half phase space  $(x^M(\sigma), p_M(\sigma))$  of the iQM. Using these representation rules we map the quantum *operators*  $O(X^M(\sigma), P_M(\sigma))$ , associated with *any conformal field theory* (CFT) on the worldsheet, to the iQM space *string field*  $O(x^M(\sigma), p_M(\sigma))$ . In particular we construct the stress tensor  $T$ , BRST current  $J_B$  and BRST operator  $Q$  as *string fields* in the half phase space,  $T(x, p)$ ,  $J_B(x, p)$ ,  $Q(x, p)$ , acting on general string fields  $A(x, p)$  only with the new Moyal star product within iQM. Then we construct the action for MSFT in this iQM formalism. The proof that the new MSFT action has a BRST gauge symmetry is given in Appendix-B.
- In section (V G) we discuss the quantum effective action for string field theory and show that it can be brought to a highly symmetric purely cubic form. The equation of motion of the purely cubic SFT is  $\bar{A}(x, p) \star \bar{A}(x, p) = 0$ . The BRST field  $Q(x, p)$  in Eq.(1.1) is

clearly a solution,  $\bar{A}_{sol}(x, p) = Q(x, p)$ , and since with our methods we can construct a  $Q(x, p)$  for all CFTs on the worldsheet, we clearly have an infinite number of solutions. Then, including fluctuations, the general field is  $\bar{A}(x, p) = Q(x, p) + g_0 A(x, p)$ . This allows us to regard the perturbative setup of string field theory in powers of  $g_0$  as the analog of a spontaneously broken version of a highly symmetric cubic theory. The last paragraphs in this section show how we can easily construct a large class of non-perturbative solutions in string field by using our methods.

- In section (VI) we give an explicit expression for the perturbative vacuum in this formalism and outline how to perform perturbative computations. We give the result of a perturbative computation for the *off-shell 3-tachyon amplitude* for the case of the CFT with a flat background. This is a test that our new methods reproduce previously computed non-trivial quantities. As already mentioned, the discussion in Appendix-A guarantees that for the CFT with a flat background there will always be agreement between computations performed with the new Moyal star product and the corresponding computations, such as amplitudes [5], performed in terms of the discrete Moyal product.
- Finally in section (VII) we give an outline of where we are heading with non-perturbative computations and physical applications of this formalism.

## II. WHAT IS THE MOYAL $\star$ PRODUCT IN MSFT?

The well known Moyal product [10] is a non-commutative associative product between any two classical functions of phase space,  $A_1(x, p) \star A_2(x, p) = A_{12}(x, p)$ , that yields a new function in phase space. The property of this product is that it reproduces one to one all products of corresponding quantum operators  $\hat{A}_1(\hat{x}, \hat{p})\hat{A}_2(\hat{x}, \hat{p}) = A_{12}(\hat{x}, \hat{p})$  in standard quantum mechanics. The Moyal product is given by

$$(A_1 \star A_2)(x, p) = A_1(x, p) \exp \left[ \frac{i\hbar}{2} \left( \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_p - \overrightarrow{\partial}_x \cdot \overleftarrow{\partial}_p \right) \right] A_2(x, p) . \quad (2.1)$$



where the left/right arrows mean differentiation of the function on the left/right sides. For practical computations it is also convenient to write it in the forms

$$(A_1 \star A_2)(x, p) = A_1 \left( \left( x' + \frac{i\hbar}{2} \frac{\overrightarrow{\partial}}{\partial p} \right), \left( p' - \frac{i\hbar}{2} \frac{\overrightarrow{\partial}}{\partial x} \right) \right) A_2(x, p), \quad (2.2)$$

$$= A_1(x', p') A_2 \left( \left( x - \frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial p'} \right), \left( p + \frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x'} \right) \right), \quad (2.3)$$

where  $(x', p')$  is set to  $(x, p)$  after differentiation. All results of quantum mechanics in the standard operator Hamiltonian formalism can be replicated in the Moyal formalism. For example, the basic phase space variables satisfy,  $x \star p = xp + \frac{i\hbar}{2}$  and  $p \star x = xp - \frac{i\hbar}{2}$ , while their star commutator is

$$[x^\mu, p_\nu]_\star = x^\mu \star p_\nu - p_\nu \star x^\mu = i\hbar \delta_\nu^\mu, \quad (2.4)$$

which is the expected result for the basic canonical quantization rules of the operators  $[\hat{x}, \hat{p}] = i\hbar$  in the Hamiltonian formalism.

In the form of Eq.(2.5) the Moyal product is valid for fermions provided the orders of fermionic  $x^M, p_M$  or fermionic  $A_1, A_2$  are respected. To include both bosons and fermions  $(x^M, p_M)$  we denote  $M = (\mu, m)$  where  $\mu$  labels bosons and  $m$  labels fermions; we also permit functions of phase space  $A(x, p)$  that could be either bosonic or fermionic. We will define  $(-1)^M$  or  $(-1)^A$  where the exponent  $M$  or  $A$  means the grade,  $M, A = 0$  for bosons and  $M, A = 1$  for fermions. Thus  $(-1)^{M \text{ or } A} = (-1)^0 = 1$  for bosons, and  $(-1)^{M \text{ or } A} = (-1)^1 = -1$  for fermions. Changing the order of factors would cost an extra minus sign when they are both fermions, such as  $\overrightarrow{\partial}_{p_M} \overleftarrow{\partial}_{x^M} = \overleftarrow{\partial}_{x^M} \overrightarrow{\partial}_{p_M} (-1)^{MN}$ . Then, the Moyal product that works for both bosons and fermions is

$$(A_1 \star A_2)(x, p) = A_1(x, p) \exp \left[ \frac{i\hbar}{2} \left( \overrightarrow{\partial}_{p_M} \overleftarrow{\partial}_{x^M} - \overleftarrow{\partial}_{p_M} \overrightarrow{\partial}_{x^M} \right) \right] A_2(x, p). \quad (2.5)$$

where the order of the derivatives and their order relative to  $A_1, A_2$  need to be respected. This gives the commutation rules (2.4) for boson as above, and also gives the anticommutation rule for fermions

$$\{x^m, p_n\}_\star = x^m \star p_n + p_n \star x^m = i\hbar \delta_n^m, \quad (2.6)$$

where we watch the orders when taking derivatives as follows,

$$A \overleftarrow{\partial}_{x^N} = (-1)^{AN} \partial_{x^N} A, \text{ or } x^M \overleftarrow{\partial}_{x^N} = (-1)^{MN} \partial_{x^N} x^M = (-1)^M \delta_N^M. \quad (2.7)$$

Then the general star commutator for any  $M = (\mu, m)$  gives

$$[x^M, p_N]_\star = i\hbar\delta_N^M, \quad (2.8)$$

for both bosons and fermions, where the symbol  $[\cdot, \cdot]_\star$  means either commutator or anti-commutator as needed. The expressions in (2.2,2.3) could be used also for both bosons and fermions.

We emphasize that the usual anticommutation rules for fermions, with both  $x^m$  and  $p_m$  Hermitian must contain  $\hbar$  rather than  $i\hbar$  on the right hand side of Eq.(2.6). In this paper, for notational uniformity of the Moyal product for both bosons and fermions, we wish to use  $i\hbar$ , so this implies that in this paper  $x^m$  for fermions is defined to be antihermitian while  $p_m$  is Hermitian.

### A. The new $\star$ product

In the Moyal formulation of SFT (MSFT), string joining, not ordinary quantum mechanics, is also represented by a stringy Moyal  $\star$  product as discovered in [2]. In this paper we suggest a new formulation of MSFT with the following new version of the  $\star$  in the  $\sigma$ -basis

$$\star = \exp \left[ \frac{i}{4} \int_0^\pi d\sigma \operatorname{sign} \left( \frac{\pi}{2} - \sigma \right) \left( \vec{\partial}_{p_M(\sigma)} \overleftarrow{\partial}_{x^M(\sigma, \varepsilon)} - \overleftarrow{\partial}_{p_M(\sigma)} \vec{\partial}_{x^M(\sigma, \varepsilon)} \right) \right]. \quad (2.9)$$

One of the most important properties of this expression is that it is independent of any background conformal field theory (CFT) that describes the free string. The reason for background independence is the fact that it is defined in phase space. The notion of phase space is quite independent of any Lagrangian formalism on the worldsheet that describes the free string propagating in some background fields. Although the Lagrangian provides a background dependent relation between velocities and momenta through the standard definition  $P_M = \partial S / (\partial_\tau X^M)$ , the properties of phase space  $(X^M, P_M)$  are completely independent of the Lagrangian. Note in particular the natural upper and lower indices: no background metric is involved in lowering the index of  $P_M$ . This is the key to background independence. Since it will require quite a few details to explain how this  $\star$  product is derived, we highlight here the crucial conceptual features of this formalism to focus the reader on aspects that are important for SFT.

1. The  $\star$  in (2.9) contains the small parameter  $\varepsilon$  in the  $x(\sigma, \varepsilon)$  which is a regulator to separate the midpoint clearly in the process of string joining. As explained later, a

regulator is crucially needed to resolve ambiguities and related associativity anomalies that were identified in the past [3][7]. This avoids the danger of formal manipulations that could lead to wrong computations, thus making the new MSFT a finite theory in which every step of a computation is well defined.

2. We define the basis on which the derivatives  $\partial_{p(\sigma)}, \partial_{x(\sigma, \varepsilon)}$  in the star product act. The string field  $A(x(\sigma, \varepsilon), p(\sigma))$  is a functional of *half of the phase space* of the string as follows. This half-phase-space  $(x(\sigma, \varepsilon), p(\sigma))$  is the regulated  $\sigma$ -basis which is closely related to the position basis  $\psi(X(\sigma, \varepsilon))$  by a Fourier transform. The regulated string position  $X(\sigma, \varepsilon)$ , which contains a small parameter  $\varepsilon$ , is related to the unregulated one  $X(\sigma)$  by a simple redefinition of the choice of independent position degrees of freedom to be used to label the string field  $\psi(X(\sigma, \varepsilon))$ . The relation is  $X(\sigma, \varepsilon) = \exp\left(-\varepsilon\sqrt{-\partial_\sigma^2}\right)X(\sigma)$  as will be explained below in some detail. The unregulated momentum  $P(\sigma)$  is the canonical conjugate to the unregulated  $X(\sigma)$ . The position and momentum degrees of freedom are split into the symmetric/antisymmetric parts relative to the midpoint at  $\sigma = \pi/2$

$$x_\pm(\sigma, \varepsilon) = \frac{1}{2}(X(\sigma, \varepsilon) \pm X(\pi - \sigma, \varepsilon)), \quad (2.10)$$

$$p_\pm(\sigma) = \frac{1}{2}(P(\sigma) \pm P(\pi - \sigma)). \quad (2.11)$$

For simplicity of notation in most of the paper we will omit the  $\pm$  on  $x_+$  and  $p_-$  (but keep the  $\pm$  for  $p_+$  and  $x_-$ ), thus we will use interchangeably the following notation

$$x_+(\sigma, \varepsilon) \equiv x(\sigma, \varepsilon) \text{ and } p_-(\sigma) \equiv p(\sigma). \quad (2.12)$$

Thus, the position space string field  $\psi(X(\sigma, \varepsilon))$  is a functional of the symmetric and antisymmetric parts of the string position  $\psi(X) = \psi(x_+, x_-)$ . The field  $A(x_+(\sigma, \varepsilon), p_-(\sigma))$  is a half-fourier transform of the field  $\psi(X) = \psi(x_+, x_-)$  in the antisymmetric variable only

$$\psi(x_+, x_-) \xrightarrow[x_- \rightarrow p_-]{\text{Fourier}} A(x_+, p_-) \quad (2.13)$$

Hence the antisymmetric  $p_-(\sigma) = -p_-(\pi - \sigma)$  is the canonical conjugate to the antisymmetric part of the string  $x_-(\sigma, \varepsilon)$  in usual first quantization if the regulator  $\varepsilon$  is ignored. Similarly, the even label  $x_+(\sigma, \varepsilon) = x_+(\pi - \sigma, \varepsilon)$  is the symmetric part of the

position, and it includes a regulated midpoint  $\bar{x}(\varepsilon) = X(\pi/2, \varepsilon) = x_+(\pi/2, \varepsilon)$ . Due to the symmetry/antisymmetry properties, it is evident that  $x_+(\sigma, \varepsilon), p_-(\sigma)$  are compatible observables: namely, their operator counterparts  $\hat{x}_+(\sigma, \varepsilon), \hat{p}_-(\sigma)$  commute with each other in the first quantization of the string, that is, although  $[\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma')] \neq 0$  does not vanish at  $\sigma = \sigma'$ , the commutator  $[\hat{x}_+(\sigma, \varepsilon), \hat{p}_-(\sigma')] = 0$  does vanish for all values of  $\sigma, \sigma'$ . Hence as eigenvalues of commuting operators,  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  is a set of complete labels for the string field  $A(x_+(\sigma, \varepsilon), p_-(\sigma))$  which is nothing but the wavefunction in first quantization taken in an appropriate basis  $A(x_+(\sigma, \varepsilon), p_-(\sigma)) = \langle x_+(\sigma, \varepsilon), p_-(\sigma) | A \rangle$ .

3. The dot products that appear in the  $\star$ , such as  $\overleftarrow{\partial}_{x(\sigma, \varepsilon)} \cdot \overrightarrow{\partial}_{p(\sigma)}$ , mean

$$\overrightarrow{\partial}_{p(\sigma)} \cdot \overleftarrow{\partial}_{x(\sigma, \varepsilon)} = \frac{\overrightarrow{\partial}}{\partial p_M(\sigma)} \frac{\overleftarrow{\partial}}{\partial x^M(\sigma, \varepsilon)}, \quad (2.14)$$

where the sum over  $M = (\mu, m)$  includes matter and ghosts  $(x^\mu, x^m)$ , where the ghosts  $x^m$  with  $m = c, b$  account for the fermionic ghost degrees of freedom  $b_{\pm\pm}(\sigma, \tau)$  and  $c^\pm(\sigma, \tau)$  as given explicitly in Eq.(5.7). This  $\star$  product applies to both bosons (label  $\mu$ ) and fermions (label  $m$ ). Changing the order of factors would cost a minus sign in the fermion-like ghost directions  $M = m$  as explained after Eq.(2.4).

4. Because no metric appears in the contraction of covariant and contravariant indices of canonical variables, the dot product in Eq.(2.14) is background independent for any set of background fields in a conformal field theory that describes the string action and the BRST operator.
5. The matter and ghosts in each supervector  $x^M, p_M$  in Eq.(2.4) can be rotated into each other under  $\text{OSp}(d|2)$  supertransformations,  $x^M \rightarrow x^N (S^{-1})^M_N$  and  $p_M \rightarrow S^N_M p_N$  where  $S \in \text{OSp}(d|2)$  mixes matter and ghost degrees of freedom. The new star product in Eq.(2.9), and therefore the string interaction terms in the new MSFT are evidently symmetric under this  $\text{OSp}(d|2)$ . Although this symmetry is broken by the BRST operator, keeping track of this symmetry greatly simplifies computations.
6. The sum over the degrees of freedom in the integral in (2.9),  $\frac{i}{4} \int_0^\pi d\sigma \text{sign}(\frac{\pi}{2} - \sigma) (\cdots)$ , may be rewritten as a half-range integral  $\frac{i}{2} \int_0^{\pi/2} d\sigma (\cdots)$  after taking into account the antisymmetric properties of  $(\cdots)$ . This shows clearly that only half of the string phase

space degrees of freedom are relevant in our formulation. We prefer the version with the full range integral because the  $\text{sign}(\frac{\pi}{2} - \sigma)$  factor will clarify several interesting features.

7. The antisymmetry of the integrand  $(\cdots)$  in (2.9) shows that the midpoint  $\bar{x}(\varepsilon) = x(\pi/2, \varepsilon)$  has no contribution to the non-trivial properties of the  $\star$  since, at the midpoint, the quantity  $(\cdots)$  vanishes due to the antisymmetry of  $\partial/\partial p_-(\sigma)$ , namely  $\partial/\partial p_-(\pi/2) = 0$ . Accordingly, the  $\star$  product (2.9) is local at the midpoint since  $\partial/\partial \bar{x}(\varepsilon)$  does not occur in it. That is, in the product  $A_1 \star A_2 = A_{12}$  the three fields  $A_1, A_2, A_{12}$  are all functions of the same midpoint  $\bar{x}(\varepsilon)$ , showing that the joining of strings is implemented *locally at the same midpoints* of the first, second and final strings. This desired property of string joining is automatically implemented in our formalism using,  $x_+^M(\sigma, \varepsilon)$  for all  $\sigma$ , without the need of a special treatment of the midpoint. This is a very important key feature that greatly simplified our formalism.

In the next section we will first review some of the properties of the standard Moyal product in quantum mechanics for a self sufficient presentation, and then indicate how to deduce from those properties that string joining is also conveniently expressed as the  $\star$  product given above.

### III. MOYAL $\star$ IN QM AS INSPIRATION FOR STRING JOINING

It is useful to recall the essential ingredients of how the Moyal product works in quantum mechanics (QM) because these same mathematical ingredients are behind the Moyal star product that describes string joining or splitting in the context of SFT [1] as discovered in [2]. We will use the same method as [2] again in this paper to construct the new  $\star$  product in the  $\sigma$ -basis and show that in this formalism MSFT may be viewed as a quantum mechanics type system which we call henceforth *induced quantum mechanics* (iQM) to distinguish it from ordinary QM.

In QM each quantum operator  $\hat{A}(\hat{x}, \hat{p})$  has a matrix representation in position space,  $\langle x_L | \hat{A} | x_R \rangle = \psi(x_L, x_R)$ , where  $x_L, x_R$  are eigenvalues of the position operator  $\hat{x}$ . All properties of the product of two operators  $\hat{A}_{12} = \hat{A}_1 \hat{A}_2$  is captured by the matrix product,  $\langle x_L | \hat{A}_{12} | x_R \rangle = \langle x_L | \hat{A}_1 \hat{A}_2 | x_R \rangle = \int dz \langle x_L | \hat{A}_1 | z \rangle \langle z | \hat{A}_2 | x_R \rangle$ , which we may write in terms of the

corresponding functions

$$\psi_{12}(x_L, x_R) = \int dz \psi_1(x_L, z) \psi_2(z, x_R). \quad (3.1)$$

This QM notation invites the reader to think of the function  $\psi(x_L, x_R)$  as an infinite dimensional matrix with continuous labels, that is associated to a quantum operator  $\hat{A}$ . Applying this notion to SFT [2], the function  $\psi(x_L, x_R)$  will be replaced by the string field in position space  $\psi(X(\sigma)) = \psi(x_L(\sigma), \bar{x}, x_R(\sigma))$ , where  $\bar{x} \equiv X(\pi/2)$  is the midpoint of the string and  $x_{L,R}(\sigma)$  are half-strings that correspond to the left/right portions of the full string relative to the midpoint,

$$x_L(\sigma) \equiv \{X(\sigma), 0 \leq \sigma < \pi/2\}, \quad x_R(\sigma) \equiv \{X(\pi - \sigma), 0 \leq \sigma < \pi/2\}. \quad (3.2)$$

The midpoint should be subtracted from  $(x_L(\sigma), x_R(\sigma))$ ; we will proceed as if this has been taken into account in order not to cloud the main idea and will return to this detail later. Then, as suggested by Witten [1], the string field will be treated like a matrix, so that the matrix product of string fields of the form

$$\psi_{12}(x_L(\sigma), \bar{x}, x_R(\sigma)) = \int Dz(\sigma) \psi_1(x_L(\sigma), \bar{x}, z(\sigma)) \psi_2(z(\sigma), \bar{x}, x_R(\sigma)), \quad (3.3)$$

corresponds to Witten's star product for computing the probability amplitude for two strings that join at their midpoints to create a new string. Note that the midpoint  $\bar{x}$  is fixed, it is a common point of the first, second and joined strings, and must be excluded in the integral  $Dz(\sigma)$ . This relates to the remarks in item 7 above about the good properties of our new  $\star$  product in Eq.(2.9) that automatically accomplishes the exclusion of the midpoint in the star product without separating it from the rest of the string as a special point.

This definition of the product among fields in SFT conveys the general idea formally for the joining of strings, but the implementation of how the matrix product (3.3) is to be carried out requires careful definition and considerable technical detail. Most current practitioners in SFT [11]-[22] implement the star product by performing computations in conformal field theory, as it was done historically in the initial computations [1][11]. By contrast, in MSFT computations are performed using only the Moyal  $\star$  where it becomes a straightforward algebraic computation without ever needing the complicated gymnastics of conformal maps. This was demonstrated in the past [3][4][5] but it becomes considerably simpler and transparent in the new formalism.

To see how to convert the matrix product to the Moyal product we go back to QM. Each operator constructed from the basic canonical conjugate operators  $(\hat{x}, \hat{p})$  has a classical image  $A(x, p)$  assembled as follows. First consider the matrix elements  $\langle x_L | \hat{A} | x_R \rangle = \psi(x_L, x_R)$  as above. Then define the classical phase space image  $A(x, p)$  of the operator  $\hat{A}$  by taking half Fourier transform as follows

$$A(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{2ipy} \psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{2ipy} \langle x + y | \hat{A} | x - y \rangle \quad (3.4)$$

where we have rewritten the function  $\psi(x_L, x_R)$  in terms of  $(x, y)$  which are the symmetric and antisymmetric combinations of  $x_L, x_R$ ,

$$\psi(x, y) \equiv \psi(x_L, x_R) = \langle x_L | \hat{A} | x_R \rangle, \quad x = \frac{x_L + x_R}{2}, \quad y = \frac{x_L - x_R}{2}. \quad (3.5)$$

The operator  $\hat{A}$  that appears in Eqs.(3.4,3.5) can be reconstructed from its classical image  $A(x, p)$  by substituting operators  $(\hat{x}, \hat{p})$  instead of the classical phase space  $(x, p)$  as follows,  $\hat{A}(\hat{x}, \hat{p}) = \int dx dp A(x, p) \{ \delta(x - \hat{x}) \delta(p - \hat{p}) \}$ , provided some operator ordering prescription for the delta functions is given. The prescription given by Weyl [10], which insures that  $\hat{A}(\hat{x}, \hat{p})$  is a Hermitian operator, is to replace the delta functions by their integral representations such that the operators  $(\hat{x}, \hat{p})$  appear in the same exponential as follows

$$\hat{A}(\hat{x}, \hat{p}) = \frac{1}{(2\pi)^2} \int dx' dp' dx dp A(x, p) \exp(ip'(x - \hat{x}) - ix'(p - \hat{p})). \quad (3.6)$$

It can be checked that Eqs.(3.4,3.6) are consistent with each other by inserting the operator (3.6) back into Eq.(3.4), computing the matrix element in position space by using  $\langle x_L | e^{ix'\hat{p} - ip'\hat{x}} | x_R \rangle = e^{ix'p'/2} e^{-ip'x_R} \delta(x' - x_L + x_R)$ , and performing the integrals. This shows that the classical function  $A(x, p)$  is the same in both Eqs.(3.4,3.6).

Now that we have a one to one correspondence between quantum operators and their classical images, we can ask the following question: if we compute the product of two operators in QM to obtain a new one  $\hat{A}_{12} = \hat{A}_1 \hat{A}_2$ , what is the rule for computing the phase space function  $A_{12}(x, p)$  which is the image of  $\hat{A}_{12}$  from the images  $A_1(x, p)$  and  $A_2(x, p)$ ? The answer to this question is the Moyal product, namely  $A_{12}(x, p) = A_1(x, p) \star A_2(x, p)$  where the  $\star$  product is defined in Eq.(2.5). Recalling that the matrix elements in position space  $(\psi_1, \psi_2, \psi_{12})(x_L, x_R)$  also reproduce the operator product, this means that the matrix product in Eq.(3.1) is also equivalent to the Moyal product, provided each function  $(\psi_1, \psi_2, \psi_{12})(x_L, x_R)$  is related to its half-Fourier transform  $(A_1, A_2, A_{12})(x, p)$  according to the prescription in Eqs.(3.4,3.5).

Coming back to SFT, the content of the previous paragraph was the basic inspiration in [2] that led to the rewriting of the matrix-like product in Eq.(3.3) as a Moyal product. In [2] this was done for the modes of the string in a flat target spacetime. Using Eq.(2.9) we now do it in the  $\sigma$ -basis which can be used in the presence of any set of background fields in a conformal field theory that describes a string. So, imitating Eq.(2.5) we are led to its stringy analog in (2.9) in order to reproduce the *matrix product in string joining*

$$A_{12}(x_+(\sigma, \varepsilon), p_-(\sigma)) = A_1(x_+(\sigma, \varepsilon), p_-(\sigma)) \star A_2(x_+(\sigma, \varepsilon), p_-(\sigma)), \quad (3.7)$$

where we replace the pointlike  $(x, p)$  in Eq.(2.5) by the stringlike  $(x_+(\sigma, \varepsilon), p_-(\sigma))$ , including the regulator  $\varepsilon$ . For now ignore the complication of the midpoint in Eq.(3.3) which is what the  $\varepsilon$  is for; we will explain this below. The basis  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  emerges from taking into account the map between  $\psi$  and  $A$  in Eq.(3.4,3.5). In this map we must take  $x_+(\sigma, \varepsilon) = \frac{1}{2}(x_L(\sigma, \varepsilon) + x_R(\sigma, \varepsilon))$ , while  $p_-(\sigma)$  should be the Fourier transform parameter for  $x_-(\sigma, \varepsilon) = \frac{1}{2}(x_L(\sigma, \varepsilon) - x_R(\sigma, \varepsilon))$ . This means  $x_+(\sigma, \varepsilon)$  is the symmetric part of the string  $X(\sigma, \varepsilon)$ , while  $p_-(\sigma)$  is the antisymmetric part of the momentum  $P(\sigma)$ . So, in terms of the full string degrees of freedom  $(X(\sigma, \varepsilon), P(\sigma))$  the relevant symmetric/antisymmetric parts are given precisely by Eqs.(2.10-2.12). This explains the logic why, except for some midpoint details, string joining is represented by the Moyal star product. We will discuss the details of the midpoint, but for now note that  $x_+(\sigma, \varepsilon)$  includes the midpoint but the string joining Moyal star excludes it as outlined in item (7) in section (II A). So no special treatment of the midpoint is needed.

It is important to emphasize that the new formalism applies now to string theory for any set of conformally consistent background fields contained in the worldsheet CFT. This is because the  $\star$  product is expressed in terms of the phase space degrees of freedom which is a notation that is independent of the geometrical details of the background fields. The information about the background geometry is buried in the canonical formalism that includes the relation between the velocity and momentum as well as in the stress tensor or BRST operator of the CFT. But none of this geometrical information enters in the star product (2.9). Hence the star product for string joining defined in this way provides a background independent method of expressing string-string interactions via the joining of strings.



### A. String Joining iQM versus QM

Although the setup presented above for the Moyal  $\star$  in MSFT resembles quantum mechanics (QM), the stringy  $\star$  in Eq.(2.9) does not arise because of the first quantization of the string as was the case of the particle in Eq.(2.5). Instead, the star product (2.9) is designed to formulate string joining or splitting. The resemblance to QM is intriguing and it even invites the thought that string joining could be considered as an origin for QM as we comment at the end of this section.

Below we will use the induced quantum mechanics (iQM) basis  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  to construct a representation of the first quantized QM operators of the string  $(\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma))$ . That is, QM will be built from iQM. This representation will clarify the relation and the difference between QM and iQM while also show how to represent all string theory operators, such as the stress tensor, BRST current  $J_B(\sigma)$  and the BRST operator, in the iQM basis  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  for any conformal field theory on the worldsheet (CFT) that describes the string moving in a conformally consistent set of background fields.

Starting from the definitions of  $x_\pm, p_\pm$  in Eq.(2.10), we write the full string first quantized QM operators  $\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)$  in terms of the antisymmetric and symmetric parts

$$\hat{X}^M(\sigma, \varepsilon) = \hat{x}_+^M(\sigma, \varepsilon) + \hat{x}_-^M(\sigma, \varepsilon), \quad \hat{P}_M(\sigma) = \hat{p}_{+M}(\sigma) + \hat{p}_{-M}(\sigma), \quad (3.8)$$

The regulator  $\varepsilon$  will be carefully discussed in section IV but it is not needed to explain the concepts in this section. Although we will carry on the regulator  $\varepsilon$  for consistency with the rest of the paper, the reader would probably understand this section more easily by first setting  $\varepsilon = 0$  everywhere, and then reviewing it again by recalling the definition of the regulated position operator,  $\hat{X}^M(\sigma, \varepsilon) = \exp(-\varepsilon\sqrt{-\partial_\sigma^2})\hat{X}^M(\sigma)$ , and that  $M = (\mu, m)$  denotes  $\mu$  for spacetime and  $m$  for ghosts.

The QM rules for the first quantization of the string are (anticommutator for the ghosts)

$$[\hat{X}^M(\sigma, \varepsilon), \hat{P}_N(\sigma')\} = [e^{-\varepsilon|\partial_\sigma|}\hat{X}^M(\sigma), \hat{P}_N(\sigma')\} = i\delta_N^M\delta_\varepsilon(\sigma, \sigma'), \quad (3.9)$$

where  $|\partial_\sigma| \equiv \sqrt{-\partial_\sigma^2}$  and

$$\delta_\varepsilon(\sigma, \sigma') \equiv e^{-\varepsilon|\partial_\sigma|}\delta(\sigma, \sigma') \quad (3.10)$$

is a regulated delta function defined below. Here  $P_M(\sigma)$  is defined for any conformal field theory, with arbitrary string background fields, in the canonical way, from the string Lagrangian,  $P_M(\sigma) = \partial L_{string}/\partial(\partial_\tau X^M(\sigma))$ . Note that, for any set of background fields in

the CFT  $L_{string}$ , the position  $X^M(\sigma)$  is a contravariant vector while the momentum  $P_M(\sigma)$  is a covariant vector, so that the commutation rules (3.9), with  $\delta_N^M \delta_\varepsilon(\sigma, \sigma')$  on the right hand side, are *background independent*. Furthermore, the  $\delta(\sigma, \sigma')$  is a Dirac delta function which must be consistent with Neumann or Dirichlet boundary conditions at the end points  $\sigma = 0, \pi$  and  $\sigma' = 0, \pi$  (D-branes). There are two types:  $\delta_\varepsilon^{nn}(\sigma, \sigma')$  for Neumann-Neumann, and  $\delta_\varepsilon^{dd}(\sigma, \sigma')$  for Dirichlet-Dirichlet, as given in Eqs.(4.2-4.5). Mostly we simply write  $\delta_\varepsilon(\sigma, \sigma')$  and use the appropriate version when necessary. When it becomes important to keep track of boundary condition properties in different directions  $M$ , more carefully we will write  $\delta_N^M \delta_{\varepsilon M}(\sigma, \sigma')$ , where  $\delta_{\varepsilon M}(\sigma, \sigma')$  stands for  $\delta_\varepsilon^{nn}(\sigma, \sigma')$  or  $\delta_\varepsilon^{dd}(\sigma, \sigma')$ . There will be more refinements introduced for the  $\delta(\sigma, \sigma')$ 's, including “even”  $\delta^+(\sigma, \sigma')$ , “odd”  $\delta^-(\sigma, \sigma')$ , and “midpoint”  $\hat{\delta}(\sigma, \sigma')$  versions, that include also a regulator  $\varepsilon$ , as will be shown explicitly below. Thus, D-branes enter our formalism in this way through the details of these delta functions that are sensitive to the boundary conditions applied to the ends of the string.

From (3.9) the QM rules for the operators  $\hat{x}_\pm, \hat{p}_\pm$  are derived. First note that the even degrees of freedom commute with the odd ones

$$[\hat{x}_\pm(\sigma, \varepsilon), \hat{p}_\mp(\sigma')] = 0, \quad (3.11)$$

while the non-trivial QM rules are

$$[\hat{x}_\pm(\sigma, \varepsilon), \hat{p}_\pm(\sigma')] = \frac{1}{4} \left[ e^{-\varepsilon|\partial_\sigma|} \left( \hat{X}(\sigma) \pm \hat{X}(\pi - \sigma) \right), \left( P(\sigma') \pm \hat{P}(\pi - \sigma') \right) \right] \quad (3.12)$$

$$= \frac{i}{2} e^{-\varepsilon|\partial_\sigma|} (\delta(\sigma, \sigma') \pm \delta(\pi - \sigma, \sigma')) \equiv \frac{i}{2} \delta_\varepsilon^\pm(\sigma, \sigma'). \quad (3.13)$$

The last equation defines the even and odd regulated delta functions  $\delta_\varepsilon^\pm(\sigma, \sigma')$  that will appear again later. Their explicit formulas are given in Eqs.(4.6-4.9) and their properties are illustrated for a small  $\varepsilon$  in Figs.(1,2).

The eigenvalues of the operator  $\hat{X}^M(\sigma, \varepsilon)$  form a complete set of labels for the wavefunction (string field) in position space  $\psi(X(\sigma, \varepsilon)) \equiv \psi(x_+(\sigma, \varepsilon), x_-(\sigma, \varepsilon))$ . This labelling is equivalent to the position space labelling without a regulator, namely  $\Psi(X(\sigma)) = \psi(X(\sigma, \varepsilon))$ , but as independent labels we prefer to use the eigenvalues of the regulated  $X(\sigma, \varepsilon)$  which amounts to a reshuffling of the eigenvalues of the unregulated  $X(\sigma)$ . Either way, the unregulated operator  $\hat{P}(\sigma)$  is represented in position space by the unregulated derivative  $\hat{P}(\sigma) \rightarrow -i\partial/\partial X(\sigma)$ . However, since we insist on the regulated labeling we must

use the chain rule to compute it on  $\psi(X(\sigma, \varepsilon))$ , that is

$$\begin{aligned}
i\hat{P}_M(\sigma)\psi(X(\cdot, \varepsilon)) &= \frac{\partial\psi(X(\cdot, \varepsilon))}{\partial X^M(\sigma)} = \int d\sigma' \frac{\partial\psi(X(\cdot, \varepsilon))}{\partial X^N(\sigma', \varepsilon)} \frac{\partial X^N(\sigma', \varepsilon)}{\partial X^M(\sigma)} \\
&= \int d\sigma' \frac{\partial\psi(X(\cdot, \varepsilon))}{\partial X^M(\sigma', \varepsilon)} e^{-\varepsilon|\partial_{\sigma'}|} \delta(\sigma, \sigma') \\
&= e^{-\varepsilon|\partial_{\sigma}|} \frac{\partial\psi(X(\cdot, \varepsilon))}{\partial X^M(\sigma, \varepsilon)}.
\end{aligned} \tag{3.14}$$

Here we have used the definition of  $X(\sigma', \varepsilon)$ , and the natural differential rule for the unregulated symbols  $\partial X(\sigma')/\partial X(\sigma) = \delta(\sigma, \sigma')$ , to obtain the following rules of computation.

$$\begin{aligned}
\text{regulated delta: } \frac{\partial X(\sigma', \varepsilon)}{\partial X(\sigma)} &= e^{-\varepsilon|\partial_{\sigma}|} \delta(\sigma, \sigma') \equiv \delta_{\varepsilon}(\sigma, \sigma'), \\
\text{unregulated delta: } \frac{\partial X(\sigma', \varepsilon)}{\partial X(\sigma, \varepsilon)} &= \delta(\sigma, \sigma'), \\
\text{regulated delta: } e^{-\varepsilon|\partial_{\sigma}|} \frac{\partial X(\sigma', \varepsilon)}{\partial X(\sigma, \varepsilon)} &= e^{-\varepsilon|\partial_{\sigma}|} \delta(\sigma, \sigma') \equiv \delta_{\varepsilon}(\sigma, \sigma'),
\end{aligned} \tag{3.15}$$

The last one, which is the only rule needed to represent the momentum according to Eq.(3.14), always yields well defined regulated results. So, when  $\psi$  is just  $X(\sigma', \varepsilon)$ , we get  $\hat{P}(\sigma)\psi = -ie^{-\varepsilon|\partial_{\sigma}|} \frac{\partial X(\sigma', \varepsilon)}{\partial X(\sigma, \varepsilon)} = -i\delta_{\varepsilon}(\sigma, \sigma')$ , which is consistent with the commutation rules (3.9). Rewriting this in terms of the symmetric/antisymmetric labels we get the representation for  $\hat{p}_{\pm}$

$$i\hat{p}_{\pm M}(\sigma)\psi(x_+(\cdot, \varepsilon), x_-(\cdot, \varepsilon)) = \frac{1}{2}e^{-\varepsilon|\partial_{\sigma}|} \left( \frac{\partial\psi(x_+, x_-)}{\partial x_{\pm}(\sigma, \varepsilon)} \right) \tag{3.16}$$

and the rules for computation are

$$\frac{\partial x_{\pm}(\sigma', \varepsilon)}{\partial x_{\pm}(\sigma, \varepsilon)} = (\delta(\sigma, \sigma') \pm \delta(\pi - \sigma, \sigma')) \equiv \delta^{\pm}(\sigma, \sigma'), \text{ unregulated delta,} \tag{3.17}$$

so that when  $\psi$  is just  $x_{\pm}(\sigma', \varepsilon)$ , we get  $\hat{p}_{\pm}(\sigma)\psi = -\frac{i}{2}e^{-\varepsilon|\partial_{\sigma}|} \frac{\partial x_{\pm}(\sigma', \varepsilon)}{\partial x_{\pm}(\sigma, \varepsilon)} = -i\delta_{\varepsilon}^{\pm}(\sigma, \sigma')$ , a regulated delta, which is consistent with the commutation rules (3.12).

In summary, in position space we have the following representation of the full string QM operators

$$\hat{X}^M(\sigma, \varepsilon)\psi(x_+, x_-) = (x_+^M(\sigma, \varepsilon) + x_-^M(\sigma, \varepsilon))\psi(x_+, x_-) \tag{3.18}$$

$$\hat{P}_M(\sigma)\psi(x_+, x_-) = -\frac{i}{2}e^{-\varepsilon|\partial_{\sigma}|} \left( \frac{\partial\psi(x_+, x_-)}{\partial x_+^M(\sigma, \varepsilon)} + \frac{\partial\psi(x_+, x_-)}{\partial x_-^M(\sigma, \varepsilon)} \right), \tag{3.19}$$

and to compute we use the differentiation rules in Eq.(3.17).

We now turn to the string field in the mixed phase space basis  $A(x_+, p_-)$ , which is the Fourier transform of  $\psi(x_+, x_-)$  in the  $(-)$  variable as in Eq.(2.13). The path-integral Fourier

transform of  $\psi(x_+^M(\cdot, \varepsilon), x_-^M(\cdot, \varepsilon))$  is precisely  $A(x(\cdot, \varepsilon), p(\cdot))$

$$A(x(\cdot, \varepsilon), p(\cdot)) = \int (Dx_-(\cdot, \varepsilon)) \exp \left[ -i \int_0^\pi d\sigma x_-^M(\sigma, \varepsilon) p_M(\sigma) \right] \psi(x_+(\cdot, \varepsilon), x_-(\cdot, \varepsilon)) \quad (3.20)$$

In the Fourier exponent we divided by a factor of 2 as compared to Eq.(3.4) to take into account the double counting due to the antisymmetry of  $x_-(\sigma, \varepsilon) \cdot p(\sigma)$  when reflected from  $\pi/2$ . In the basis  $A(x_+, p_-)$  the operators  $\hat{x}_+(\sigma, \varepsilon), \hat{p}_-(\sigma)$  are diagonal, while the operators  $\hat{p}_+, \hat{x}_-$  act like derivatives. Either by taking Fourier transform of Eqs.(3.18,3.19) or directly from the commutation rules (3.12) we derive the representations of all operators  $\hat{x}_\pm, \hat{p}_\pm$  on this basis

$$\hat{x}_+(\sigma, \varepsilon) A(x_+, p_-) = x_+(\sigma, \varepsilon) A(x_+, p_-), \quad \hat{p}_-(\sigma) A(x_+, p_-) = e^{-\varepsilon|\partial_\sigma|} p_-(\sigma) A(x_+, p_-) \quad (3.21)$$

$$\hat{x}_-^M(\sigma, \varepsilon) A(x_+, p_-) = \frac{i}{2} \frac{\partial A(x_+, p_-)}{\partial p_{-M}(\sigma)}, \quad \hat{p}_{+M}(\sigma) A(x_+, p_-) = -\frac{i}{2} e^{-\varepsilon|\partial_\sigma|} \frac{\partial A(x_+, p_-)}{\partial x_+^M(\sigma, \varepsilon)}. \quad (3.22)$$

Then the full string QM operators have the following representation

$$\hat{X}^M(\sigma, \varepsilon) A(x_+, p_-) = \left( x_+^M(\sigma, \varepsilon) + \frac{i}{2} \frac{\partial}{\partial p_{-M}(\sigma)} \right) A(x_+, p_-), \quad (3.23)$$

$$\hat{P}_M(\sigma) \psi(x_+, x_-) = e^{-\varepsilon|\partial_\sigma|} \left( p_{-M}(\sigma) - \frac{i}{2} \frac{\partial}{\partial x_+^M(\sigma, \varepsilon)} \right) A(x_+, p_-). \quad (3.24)$$

From Eq.(3.9) to Eq.(3.24) we described the QM of the string in any background CFT and how to represent its basic quantum operators  $\hat{X}, \hat{P}$  on the field  $A(x_+, p_-)$ . This is sufficient to obtain the representation of any other observable in this CFT, such as the BRST operator, in the MSFT formalism. We will do this later, but first we relate this differential operator representation to something more elegant in the language of the Moyal  $\star$  product.

Now we turn to the induced QM (iQM) generated by the Moyal  $\star$  for string joining for any CFT. We will show that QM is represented in terms of iQM. First observe that using the  $\star$  in Eq.(2.9), and the differentiation rules (3.17), we get a non-trivial  $\star$  commutator

between  $x_+(\sigma, \varepsilon)$  and  $p_-(\sigma)$

$$\begin{aligned}
& [x_+^M(\sigma_1, \varepsilon), p_{-N}(\sigma_2)]_\star \\
&= x_+^M(\sigma_1, \varepsilon) \star p_{-N}(\sigma_2) - (-1)^{MN} p_{-N}(\sigma_2) \star x_+^M(\sigma_1, \varepsilon) \\
&= i\delta_N^M \operatorname{sign}\left(\frac{\pi}{2} - \sigma_1\right) \delta^-(\sigma_1, \sigma_2) = i\delta_N^M \delta^+(\sigma_1, \sigma_2) \operatorname{sign}\left(\frac{\pi}{2} - \sigma_2\right) \\
&\equiv i\delta_N^M \hat{\delta}_{+-}(\sigma_1, \sigma_2).
\end{aligned} \tag{3.25}$$

This is the induced QM. It has nothing to do with ordinary quantum mechanics since under QM the operators  $\hat{x}_+^M(\sigma_1, \varepsilon)$ ,  $\hat{p}_{-N}(\sigma_2)$  commute with each other as seen in Eq.(3.11), whereas their eigenvalues do not commute under the string-joining iQM of Eq.(3.25).

The Dirac  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  function, that combines the sign function with the unregulated delta functions  $\delta^\pm$  of Eq.(3.13), satisfies the appropriate  $(nn)$  or  $(dd)$  boundary conditions consistent with the properties of  $x_+^M(\sigma_1, \varepsilon)$ ,  $p_{-N}(\sigma_2)$ . Furthermore, relative to the midpoint  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  is symmetric in  $\sigma_1 \rightarrow (\pi - \sigma_1)$  and antisymmetric in  $\sigma_2 \rightarrow (\pi - \sigma_2)$ . The equivalence of the two forms of  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  given above is verified by using the properties of the unregulated delta functions as follows

$$\begin{aligned}
\hat{\delta}_{+-}(\sigma_1, \sigma_2) &= \operatorname{sign}\left(\frac{\pi}{2} - \sigma_1\right) \delta^-(\sigma_1, \sigma_2) \\
&= \operatorname{sign}\left(\frac{\pi}{2} - \sigma_1\right) (\delta(\sigma_1, \sigma_2) - \delta(\sigma_1, \pi - \sigma_2)) \\
&= \operatorname{sign}\left(\frac{\pi}{2} - \sigma_2\right) \delta(\sigma_1, \sigma_2) - \operatorname{sign}\left(\frac{\pi}{2} - (\pi - \sigma_2)\right) \delta(\sigma_1, \pi - \sigma_2) \\
&= \operatorname{sign}\left(\frac{\pi}{2} - \sigma_2\right) \delta(\sigma_1, \sigma_2) - \operatorname{sign}\left(-\left(\frac{\pi}{2} - \sigma_2\right)\right) \delta(\sigma_1, \pi - \sigma_2) \\
&= (\delta(\sigma_1, \sigma_2) + \delta(\sigma_1, \pi - \sigma_2)) \operatorname{sign}\left(\frac{\pi}{2} - \sigma_2\right) \\
&= \delta^+(\sigma_1, \sigma_2) \operatorname{sign}\left(\frac{\pi}{2} - \sigma_2\right).
\end{aligned} \tag{3.26}$$

Thus,  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  is a distribution that has the following important properties under reflections from the midpoint: it is even under  $\sigma_1 \rightarrow (\pi - \sigma_1)$  and odd under  $\sigma_2 \rightarrow (\pi - \sigma_2)$  and vanishes at both  $\sigma_1 = \pi/2$  and  $\sigma_2 = \pi/2$ . These properties can be verified from the two equivalent forms of  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  given above, or by integrating with smooth functions, or by expanding in modes. This means in Eq.(3.25) that the midpoint  $\bar{x}^M(\varepsilon) = x_+^M(\pi/2, \varepsilon)$  star-commutes with  $p_{-N}(\sigma_2)$  including  $\sigma_2 = \pi/2$ , so the midpoint does not participate in the joining operation of strings - this is one of the desired properties of the  $\star$  product as emphasized in point 7 in section (II A). This vital property is encoded in the  $\star$  product in Eq.(2.9) as well as the  $\hat{\delta}_{+-}(\sigma_1, \sigma_2)$  that appears in the iQM.

Central features of the iQM follows from the following left and right products of the field with the classical half phase space  $x_+(\sigma, \varepsilon), p_-(\sigma)$

$$x_+^M(\sigma, \varepsilon) \star A(x_+, p_-) = \left( x_+^M(\sigma, \varepsilon) + \frac{i}{2} \text{sign}\left(\frac{\pi}{2} - \sigma\right) \partial_{p_-(\sigma)} \right) A(x_+, p_-) \quad (3.28)$$

$$A(x_+, p_-) \star x_+^M(\sigma, \varepsilon) = A(x_+, p_-) \left( x_+^M(\sigma, \varepsilon) - \frac{i}{2} \overleftarrow{\partial}_{p_-(\sigma)} \text{sign}\left(\frac{\pi}{2} - \sigma\right) \right) \quad (3.29)$$

$$p_-(\sigma) \star A(x_+, p_-) = \left( p_-(\sigma) - \frac{i}{2} \text{sign}\left(\frac{\pi}{2} - \sigma\right) \partial_{x_+^M(\sigma, \varepsilon)} \right) A(x_+, p_-) \quad (3.30)$$

$$A(x_+, p_-) \star p_-(\sigma) = A(x_+, p_-) \left( p_-(\sigma) + \frac{i}{2} \overleftarrow{\partial}_{x_+^M(\sigma, \varepsilon)} \text{sign}\left(\frac{\pi}{2} - \sigma\right) \right) \quad (3.31)$$

By comparing these results to Eqs.(3.23,3.24) we see that the full string QM operators  $\hat{X}^M(\sigma, \varepsilon), \hat{P}_M(\sigma)$  can be represented in terms of the string joining star product in the half phase space as follows

$$\hat{X}^M(\sigma, \varepsilon) A(x_+, p_-) = \left[ \begin{array}{l} \theta\left(\frac{\pi}{2} - \sigma\right) x_+^M(\sigma, \varepsilon) \star A(x_+, p_-) \\ + \theta\left(\sigma - \frac{\pi}{2}\right) A(x_+, p_-) \star x_+^M(\sigma, \varepsilon) (-1)^{MA} \end{array} \right], \quad (3.32)$$

$$\hat{P}_M(\sigma) A(x_+, p_-) = e^{-\varepsilon|\partial_\sigma|} \left[ \begin{array}{l} \theta\left(\frac{\pi}{2} - \sigma\right) p_-(\sigma) \star A(x_+, p_-) \\ + \theta\left(\sigma - \frac{\pi}{2}\right) A(x_+, p_-) \star p_-(\sigma) (-1)^{MA} \end{array} \right], \quad (3.33)$$

where the sign factor  $(-1)^{MA}$  accounts for boson/fermion properties of the field  $A$  and the operator labeled by  $M$ . Described in words, the structures (3.32,3.33) show that depending on whether  $\sigma$  is less or more than  $\pi/2$  the action of the full string quantum operators  $\hat{X}^M(\sigma, \varepsilon), \hat{P}_M(\sigma)$  on the field is reproduced by the left or right Moyal  $\star$  product with the *half phase space*. In more detail, one can check that the star product reproduces the non-derivative as well the derivative terms in Eqs.(3.23,3.24) for all  $0 \leq \sigma \leq \pi$ , including  $\sigma = \pi/2$ , since

$$1 = \theta\left(\frac{\pi}{2} - \sigma\right) + \theta\left(\sigma - \frac{\pi}{2}\right), \quad (3.34)$$

$$1 = \text{sign}\left(\frac{\pi}{2} - \sigma\right) \left( \theta\left(\frac{\pi}{2} - \sigma\right) - \theta\left(\sigma - \frac{\pi}{2}\right) \right). \quad (3.35)$$

It is clear that this works correctly as long as  $\sigma \neq \pi/2$ . In order to also work correctly at  $\sigma = \pi/2$  we must define carefully what values the symbols  $\text{sign}(\frac{\pi}{2} - \sigma), \theta(\frac{\pi}{2} - \sigma), \theta(\sigma - \frac{\pi}{2})$  take at  $\sigma = \pi/2$ . Thus, in our *definition*, the distribution  $\text{sign}(\frac{\pi}{2} - \sigma)$  does not vanish at  $\sigma = \pi/2$ , but rather its value at  $\pi/2$  is  $\pm 1$  depending on the approach to the midpoint from below or above as  $\sigma \rightarrow \pi/2 \mp 0$ . Similarly, in our definitions, the functions  $\theta(\frac{\pi}{2} - \sigma)$

or  $\theta(\sigma - \frac{\pi}{2})$  do not take the value  $1/2$  at  $\sigma = \pi/2$ , but rather they are equal to 1 or 0 depending on the approach to the midpoint from below or above as  $\sigma \rightarrow \pi/2 \mp 0$ . Then,  $\text{sign}(\frac{\pi}{2} - \sigma) = \theta(\frac{\pi}{2} - \sigma) - \theta(\sigma - \frac{\pi}{2})$ , takes the values  $\pm 1$  as usual, except that this does not vanish at  $\pi/2$  due to the careful definition. Hence Eq.(3.35) is satisfied for all  $0 \leq \sigma \leq \pi$ , including  $\sigma = \pi/2$ .

One can now check that all the rules of quantum mechanics from Eq.(3.9) to Eq.(3.24) are correctly reproduced by the iQM representation of the operators (3.32,3.33), including at the midpoint  $\sigma = \pi/2$ . From now on we do not need anymore the  $\pm$  labels on the  $x_+, p_-$  and we can write the representation of the quantum operators more simply as

$$\hat{X}^M(\sigma, \varepsilon) A(x, p) = \begin{cases} x^M(\sigma, \varepsilon) \star A(x, p), & \text{if } 0 \leq \sigma \leq \pi/2 \\ A(x, p) \star x^M(\sigma, \varepsilon) (-1)^{MA}, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}, \quad (3.36)$$

$$\hat{P}_M(\sigma) A(x, p) = \begin{cases} (e^{-\varepsilon|\partial_\sigma|} p_M(\sigma)) \star A(x, p), & \text{if } 0 \leq \sigma \leq \pi/2 \\ A(x, p) \star (e^{-\varepsilon|\partial_\sigma|} p_M(\sigma)) (-1)^{MA}, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}. \quad (3.37)$$

We also do not need to watch too carefully the midpoint  $\bar{x}(\varepsilon)$  in most cases since we have seen that  $\bar{x}(\varepsilon)$  acts trivially (like a number, or an eigenvalue) under the  $\star$  in iQM. If need be, at  $\sigma = \pi/2$  we can use Eqs.(3.32,3.33) or equivalently Eqs.(3.23,3.24) if more care is warranted in some computations.

Indeed, note that the  $\star$  in iQM in Eq.(3.33) does produce correctly a derivative contribution of the momentum operator  $\hat{P}_M(\sigma)$  at the midpoint with a regulator  $\hat{P}_M(\pi/2) \rightarrow (-i/2) e^{-\varepsilon|\partial_{\pi/2}|} (\partial/\partial x(\pi/2, \varepsilon))$ , as it should be, according to QM in Eq.(3.24). This is a bit subtle and requires more explanation. Having pointed out earlier that the derivatives in the star product do not act on the midpoint when considering string joining, one may wonder how the midpoint derivative in Eq.(3.24) is reproduced from the star product representation in Eq.(3.33). This subtle point is explained as follows. Consider evaluating the star products in (3.33) by using (3.30,3.31) and concentrate on the derivative piece which takes the form

$$- \frac{i}{4} e^{-\varepsilon|\partial_\sigma|} \left[ \left( \theta\left(\frac{\pi}{2} - \sigma\right) - \theta\left(\sigma - \frac{\pi}{2}\right) \right) \int_0^\pi d\sigma' \delta^-(\sigma, \sigma') \text{sign}\left(\frac{\pi}{2} - \sigma'\right) \frac{\partial A}{\partial x_+(\sigma', \varepsilon)} \right], \quad (3.38)$$

where the delta function arises from  $\partial p_-(\sigma)/\partial p_-(\sigma') = \delta^-(\sigma, \sigma')$ . Naively this expression appears to vanish at  $\sigma = \pi/2$  since  $\delta^-(\sigma, \sigma')$  vanishes at the midpoint, and hence no midpoint contribution; but more care is needed. The distribution in the integrand has the following property according to Eqs.(3.26-3.27),  $\delta^-(\sigma, \sigma') \text{sign}(\frac{\pi}{2} - \sigma') = \text{sign}(\frac{\pi}{2} - \sigma) \delta^+(\sigma, \sigma')$ . Both

of these forms vanish at the midpoint as argued in (3.26-3.27). However, using the second form, the sign function  $\text{sign}(\frac{\pi}{2} - \sigma)$  can be pulled out of the  $d\sigma'$  integral, and after combining it with the theta function factor in (3.38) it gives an overall factor of 1 according to Eq.(3.35), including at the midpoint (we emphasize the careful definition of the sign function). In the remaining  $d\sigma'$  integrand  $\delta^+(\sigma, \sigma')$  by itself does not vanish at the midpoint, and Eq.(3.38) yields the following result (recall that  $\delta^+(\sigma, \sigma')$  has two peaks in the range  $[0, \pi]$ )

$$-\frac{i}{4}e^{-\varepsilon|\partial_\sigma|} \left[ \int_0^\pi d\sigma' \delta^+(\sigma, \sigma') \frac{\partial A}{\partial x_+(\sigma', \varepsilon)} \right] = -\frac{i}{2}e^{-\varepsilon|\partial_\sigma|} \frac{\partial A}{\partial x_+(\sigma, \varepsilon)}. \quad (3.39)$$

This is consistent with the expected result in Eq.(3.24) which includes the midpoint derivative. The subtle property here is that the distribution  $\text{sign}(\frac{\pi}{2} - \sigma) \delta^+(\sigma, \sigma') = \delta^-(\sigma, \sigma') \text{sign}(\frac{\pi}{2} - \sigma')$  has no support at the midpoint, but  $\delta^+(\sigma, \sigma')$  by itself does. The theta function factor was crucial to remove the sign factor  $\text{sign}(\frac{\pi}{2} - \sigma)$  and lead the non-trivial midpoint contribution. This exercise makes it evident that there are circumstances in some computations where midpoint derivatives can arise from the new star product, and this is indeed desirable, although straightforward generic string joining  $A \star B$  is not one of those circumstances.

#### IV. REGULATOR

Consider the fundamental canonical QM operators in string theory  $(\hat{X}^M(\sigma), \hat{P}_M(\sigma))$  at fixed  $\tau$  for any CFT on the worldsheet. A basic tool of computation in CFT is the operator product expansion which is a form of regularization that controls operator products at the same point on the worldsheet. A moment of reflection would indicate that the same regularization effect is captured by our proposed choice of independent degrees of freedom  $(\hat{X}^M(\sigma, \varepsilon), \hat{P}_M(\sigma))$  where  $\hat{X}^M(\sigma, \varepsilon) = e^{-\varepsilon|\partial_\sigma|} \hat{X}^M(\sigma)$  is regulated while  $\hat{P}_M(\sigma)$  is not. This is because  $|\partial_\sigma| \equiv \sqrt{-\partial_\sigma^2}$  plays the role of an approximate time translation operator on the worldsheet for a short amount of time even when there are background fields present in the CFT. Thus applying a Euclidean time translation in the form  $e^{-\varepsilon|\partial_\sigma|} \hat{X}^M(\sigma)$  displaces the worldsheet point from  $(\sigma, \tau = 0)$  to  $(\sigma, \tau = -i\varepsilon)$ . Equivalently a point on the unit circle in the complex  $z$  plane ( $z \equiv e^{\pm i\sigma}$ ) moves to the inside of the unit circle when the Euclidean time translator  $e^{-\varepsilon|\partial_\sigma|}$  acts on it, namely  $e^{-\varepsilon|\partial_\sigma|} z = e^{-\varepsilon|\partial_\sigma|} e^{\pm i\sigma} = e^{-\varepsilon \pm i\sigma} = \tilde{z}$ , with  $|\tilde{z}| < 1$ . In this computation we used the fact that  $e^{\pm i\sigma}$  (and similarly  $e^{\pm i n \sigma}$ ) are degenerate eigenstates



of the operator  $|\partial_\sigma|$ , that is

$$|\partial_\sigma| e^{\pm i n \sigma} = \sqrt{-\partial_\sigma^2} e^{\pm i n \sigma} = |n| e^{\pm i n \sigma}. \quad (4.1)$$

Hence changing  $\hat{X}(\sigma)$  to  $\hat{X}^M(\sigma, \varepsilon)$  amounts precisely to what is done in operator products when two points are slightly displaced relative to each other, one on the unit circle and the other inside the unit circle. By defining the theory including the regulator  $\varepsilon$  as proposed, we can control all the relevant operator products in CFT at all intermediate stages of CFT computations. Carrying this  $\varepsilon$  over to MSFT as we have done in the previous sections insures that all MSFT computations will be finite at all intermediate stages. Only at the end of MSFT computations we will set  $\varepsilon = 0$  after a renormalization of the cubic coupling constant.

### A. Regulated delta functions

To perform computations in MSFT we must use the differentiation rules, such as  $e^{-\varepsilon|\partial_{\sigma_2}|}(\partial X(\sigma_1, \varepsilon)/\partial X(\sigma_2, \varepsilon))$  etc., that emerged in section (III A) to represent the basic quantum operators  $\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)$ . The result of such derivatives involves various types of delta functions, with or without regularization, that are also sensitive to D-brane boundary conditions. In later computations there will be circumstances in which we must multiply such delta functions with each other. Products of unregulated delta functions are not well defined. However, in our case the parameter  $\varepsilon$  provides just the required regularization to render such products well defined. Since the regularized deltas will be used for computation, we provide the needed details for them below.

The basic case from which all others follow is the delta function that appears in the QM commutation rules in Eq.(3.9)  $[\hat{X}^M(\sigma_1, \varepsilon), \hat{P}_N(\sigma_2)] = i\delta_N^M \delta_\varepsilon^M(\sigma_1, \sigma_2)$ , where the  $M$  on  $\delta_\varepsilon^M(\sigma_1, \sigma_2)$  is a reminder that in the direction  $M$  the operators satisfy either Neumann-Neumann ( $nn$ ) or Dirichlet-Dirichlet ( $dd$ ) boundary conditions. Accordingly  $\delta_\varepsilon^M(\sigma_1, \sigma_2)$  will be either  $\delta_\varepsilon^{nn}(\sigma_1, \sigma_2)$  or  $\delta_\varepsilon^{dd}(\sigma_1, \sigma_2)$ .

Thus for ( $nn$ ) we have the following unique expression,  $\delta_\varepsilon^{nn}(\sigma_1, \sigma_2) = e^{-\varepsilon|\partial_{\sigma_1}|} \delta^{nn}(\sigma_1, \sigma_2)$ , that is a periodic Dirac delta function  $\delta^{nn}(\sigma_1, \sigma_2)$  (when  $\varepsilon = 0$ ) which also satisfies the boundary conditions - its *derivatives* vanish at the string ends for either  $\sigma_1 = 0, \pi$  or  $\sigma_2 = 0, \pi$ . The regulated version is computed easily since  $\cos n\sigma_1$  is an eigenstate of  $|\partial_{\sigma_1}|$ , namely

$|\partial_{\sigma_1}| \cos n\sigma_1 = \sqrt{-\partial_{\sigma_1}^2} \cos n\sigma_1 = |n| \cos n\sigma_1$ . Hence the regulated  $\delta_\varepsilon^{nn}(\sigma_1, \sigma_2)$  is

$$\delta_\varepsilon^{nn}(\sigma_1, \sigma_2) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n \geq 1} e^{-\varepsilon n} \cos n\sigma_1 \cos n\sigma_2. \quad (4.2)$$

By writing the cosines in terms of exponentials the series turns into a *convergent* geometric series for any positive  $\varepsilon$ , so that it can be summed up and written in the following exact form, and then approximated for small  $\varepsilon$

$$\begin{aligned} \delta_\varepsilon^{nn}(\sigma_1, \sigma_2) &= \frac{1}{2\pi} \left( \frac{\sinh \frac{\varepsilon}{2} \cosh \frac{\varepsilon}{2}}{\sinh^2 \frac{\varepsilon}{2} + \sin^2 \frac{\sigma_1 - \sigma_2}{2}} + \frac{\sinh \frac{\varepsilon}{2} \cosh \frac{\varepsilon}{2}}{\sinh^2 \frac{\varepsilon}{2} + \sin^2 \frac{\sigma_1 + \sigma_2}{2}} \right) \\ &\simeq \frac{\varepsilon/\pi}{\varepsilon^2 + 4 \sin^2 \frac{\sigma_1 - \sigma_2}{2}} + \frac{\varepsilon/\pi}{\varepsilon^2 + 4 \sin^2 \frac{\sigma_1 + \sigma_2}{2}} \\ &\simeq \delta \left( 2 \sin \frac{\sigma_1 - \sigma_2}{2} \right) + \delta \left( 2 \sin \left( \frac{\sigma_1 + \sigma_2}{2} \right) \right) \end{aligned} \quad (4.3)$$

Recall that only the range  $0 \leq \sigma_1, \sigma_2 \leq \pi$  matters. The first term has a peak at  $\sigma_1 = \sigma_2$  when  $\sigma_1, \sigma_2$  are both in the range  $[0, \pi]$ , while the second term has no peak in this range unless  $\sigma_1, \sigma_2$  are both at the end points 0 or  $\pi$ . For example if  $\sigma_2 = 0$  (or  $\pi$ ) both terms have peaks at  $\sigma_1 = 0$  (or  $\pi$ ), but as seen with a nonzero  $\varepsilon$ , only half of the area under each curve falls within the range  $[0, \pi]$ , so the effect is that the end points are included with the same strength as any interior point. This is what should be expected with Neumann-Neumann boundary conditions.

Similarly, for Dirichlet-Dirichlet boundary conditions  $\delta_\varepsilon^{dd}(\sigma_1, \sigma_2)$  appears in the commutation rules. Due to the D-brane boundaries it must vanish at both string ends  $\sigma_1 = 0, \pi$  and  $\sigma_2 = 0, \pi$ . Then it is uniquely given by

$$\delta_\varepsilon^{dd}(\sigma_1, \sigma_2) \equiv \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\varepsilon n} \sin n\sigma_1 \sin n\sigma_2. \quad (4.4)$$

Summing up the geometric series we obtain the exact form for any  $\varepsilon$  and approximate forms for small  $\varepsilon$

$$\begin{aligned} \delta_\varepsilon^{dd}(\sigma_1, \sigma_2) &= \frac{1}{2\pi} \left( \frac{\sinh \frac{\varepsilon}{2} \cosh \frac{\varepsilon}{2}}{\sinh^2 \frac{\varepsilon}{2} + \sin^2 \frac{\sigma_1 - \sigma_2}{2}} - \frac{\sinh \frac{\varepsilon}{2} \cosh \frac{\varepsilon}{2}}{\sinh^2 \frac{\varepsilon}{2} + \sin^2 \frac{\sigma_1 + \sigma_2}{2}} \right) \\ &\simeq \frac{\varepsilon/\pi}{\varepsilon^2 + 4 \sin^2 \frac{\sigma_1 - \sigma_2}{2}} - \frac{\varepsilon/\pi}{\varepsilon^2 + 4 \sin^2 \frac{\sigma_1 + \sigma_2}{2}} \\ &\simeq \delta \left( 2 \sin \frac{\sigma_1 - \sigma_2}{2} \right) - \delta \left( 2 \sin \left( \frac{\sigma_1 + \sigma_2}{2} \right) \right) \end{aligned} \quad (4.5)$$

The first term has a peak at  $\sigma_1 = \sigma_2$  while the second one has no peak in the range  $0 \leq \sigma_1, \sigma_2 \leq \pi$  unless  $\sigma_1, \sigma_2$  are both at the end points, but at either end point the peaks of the two terms cancel each other. So there is no support at the end points. This is what should be expected with Dirichlet-Dirichlet boundary conditions.

Now we can compute the other delta functions,  $\delta_\varepsilon^\pm(\sigma_1, \sigma_2) = \delta_\varepsilon(\sigma_1, \sigma_2) \pm \delta_\varepsilon(\sigma_1, \pi - \sigma_2)$ , either  $(nn)$  or  $(dd)$ , that emerge in taking derivatives with respect to  $x_+^M(\sigma, \varepsilon)$  or  $p_M(\sigma)$ . These are given by

$$\delta_\varepsilon^{+nn}(\sigma_1, \sigma_2) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{e \geq 2} e^{-\varepsilon e} \cos e\sigma_1 \cos e\sigma_2, \quad e = 2, 4, 6, \dots \quad (4.6)$$

$$\delta_\varepsilon^{-nn}(\sigma_1, \sigma_2) = \frac{4}{\pi} \sum_{o \geq 1} e^{-\varepsilon o} \cos o\sigma_1 \cos o\sigma_2, \quad o = 1, 3, 5, \dots \quad (4.7)$$

where  $(e, o)$  are (even, odd) positive integers. Similarly, for  $(dd)$  boundary conditions we have

$$\delta_\varepsilon^{+dd}(\sigma_1, \sigma_2) = \frac{4}{\pi} \sum_{o \geq 1} e^{-\varepsilon o} \sin o\sigma_1 \sin o\sigma_2, \quad o = 1, 3, 5, \dots \quad (4.8)$$

$$\delta_\varepsilon^{-dd}(\sigma_1, \sigma_2) = \frac{4}{\pi} \sum_{e \geq 2} e^{-\varepsilon e} \sin e\sigma_1 \sin e\sigma_2, \quad e = 2, 4, 6, \dots \quad (4.9)$$

The infinite sums can again be performed exactly. The result is obtained by applying the instruction  $\delta_\varepsilon^{\pm M}(\sigma_1, \sigma_2) = \delta_\varepsilon^M(\sigma_1, \sigma_2) \pm \delta_\varepsilon^M(\sigma_1, \pi - \sigma_2)$  to the expressions in Eqs.(4.3,4.3) for  $M = (nn)$  or  $(dd)$ . Their fully summed exact expressions for any  $\varepsilon$  are

$$\delta_\varepsilon^{+nn}(\sigma_1, \sigma_2) = \frac{2(\sinh 2\varepsilon)(\cosh 2\varepsilon - \cos 2\sigma_1 \cos 2\sigma_2)}{\pi(\cosh 2\varepsilon - \cos 2(\sigma_1 - \sigma_2))(\cosh 2\varepsilon - \cos 2(\sigma_1 + \sigma_2))} \quad (4.10)$$

$$\delta_\varepsilon^{-nn}(\sigma_1, \sigma_2) = \frac{8(\sinh \varepsilon)(\cos \sigma_1)(\cos \sigma_2)(1 + \cosh^2 \varepsilon - \cos^2 \sigma_1 - \cos^2 \sigma_2)}{\pi(\cosh 2\varepsilon - \cos 2(\sigma_1 - \sigma_2))(\cosh 2\varepsilon - \cos 2(\sigma_1 + \sigma_2))} \quad (4.11)$$

$$\delta_\varepsilon^{+dd}(\sigma_1, \sigma_2) = \frac{8(\sinh \varepsilon)(\sin \sigma_1)(\sin \sigma_2)(\cos^2 \sigma_1 + \cos^2 \sigma_2 + \sinh^2 \varepsilon)}{\pi(\cosh 2\varepsilon - \cos 2(\sigma_1 - \sigma_2))(\cosh 2\varepsilon - \cos 2(\sigma_1 + \sigma_2))} \quad (4.12)$$

$$\delta_\varepsilon^{-dd}(\sigma_1, \sigma_2) = \frac{2(\sinh 2\varepsilon)(\sin 2\sigma_1)(\sin 2\sigma_2)}{\pi(\cosh 2\varepsilon - \cos 2(\sigma_1 - \sigma_2))(\cosh 2\varepsilon - \cos 2(\sigma_1 + \sigma_2))} \quad (4.13)$$

From this we see that  $\delta_\varepsilon^{\pm M}(\sigma_1, \sigma_2)$  have two peaks in the range  $0 \leq \sigma_1, \sigma_2 \leq \pi$ , one at  $\sigma_1 = \sigma_2$  and the other at  $\sigma_1 = \pi - \sigma_2$ . For small  $\varepsilon$ , the  $(nn)$ ,  $(dd)$  distributions are almost the same for most of the range, but they differ close to the end points as seen by comparing the plots in Figs.(1,2), namely  $\delta_\varepsilon^{\pm dd}(\sigma_1, \sigma_2)$  vanishes at the end points. In the case of  $\delta_\varepsilon^{+M}(\sigma_1, \sigma_2)$

both peaks are positive, but in the case of  $\delta_\epsilon^{-M}(\sigma_1, \sigma_2)$  the second peak is negative; so when the peaks are at the midpoint  $\sigma_1 = \pi/2 = \sigma_2$ , the peaks in  $\delta_\epsilon^{+M}(\sigma_1, \sigma_2)$  add, while the peaks in  $\delta_\epsilon^{-M}(\sigma_1, \sigma_2)$  cancel each other. Finally, when the peaks are all the way at the end points,  $\delta_\epsilon^{\pm nn}(\sigma_1, \sigma_2)$  have support with half of the area under each peak at each end point, but  $\delta_\epsilon^{\pm dd}(\sigma_1, \sigma_2)$  vanishes at each end point. These properties are illustrated in Figs.(1,2) for a finite but small value of  $\epsilon$ . As  $\epsilon$  approaches zero the peaks become very tall and very narrow while the plots for  $(nn)$  and  $(dd)$  appear to converge to each other and become the same. But they are actually different from each other exactly at the end points even when  $\epsilon = 0$ .

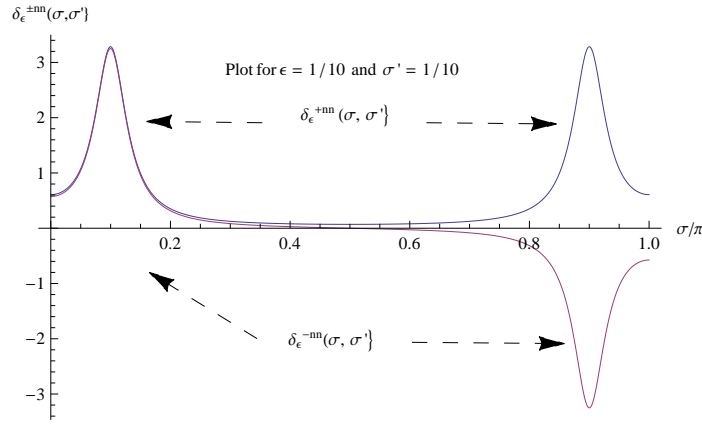


Fig.(1) - Plot of  $\delta_\epsilon^{\pm nn}(\sigma, \sigma')$  for  $\epsilon = 1/10$  and  $\sigma' = 1/10$

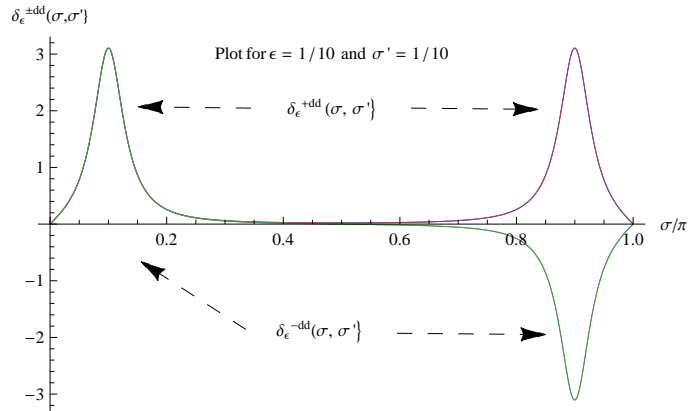


Fig.(2) - Plot of  $\delta_\epsilon^{\pm dd}(\sigma, \sigma')$  for  $\epsilon = 1/10$  and  $\sigma' = 1/10$

## B. Midpoint not treated separately

We have seen that the new  $\star$  product (2.9) does not treat the midpoint in a special way, nevertheless is able to subtly exclude the midpoint from the star product in the process of

string joining. This desirable outcome seems natural but it took a lot of effort to reach this stage: after going through many several alternative formalisms in which the midpoint was explicitly separated as suggested in the discussion after Eq.(3.3), we eventually discovered that there is a better way of choosing the independent variables to label the string field, as presented above, and *then* the midpoint need not be treated separately from the rest.

To clarify this point let us show how one could setup a formalism in which the midpoint is treated differently from the rest. The regulated delta functions are very useful to provide the following well defined separation

$$x_+(\sigma, \varepsilon) = \tilde{x}(\sigma, \varepsilon) + \frac{\delta_\varepsilon(\sigma, \pi/2)}{\delta_\varepsilon(0)} \bar{x}(\varepsilon); \quad \delta_\varepsilon(0) \equiv \delta_\varepsilon(\pi/2, \pi/2) \simeq \frac{1}{\pi\varepsilon} \quad (4.14)$$

where the midpoint is  $\bar{x}(\varepsilon) = x(\pi/2, \varepsilon)$ . The  $\tilde{x}(\sigma, \varepsilon)$  which vanishes at the midpoint,  $\tilde{x}(\pi/2, \varepsilon) = 0$ , is the rest of the symmetric  $x_+(\sigma, \varepsilon)$ . Then treating  $\tilde{x}(\sigma, \varepsilon), \bar{x}(\varepsilon)$  as independent variables, and using the chain rule, we construct the derivative representation of the canonical variable  $\hat{P}_+(\sigma)$  in Eq.(3.16)

$$\hat{P}_+(\sigma) \rightarrow \frac{-i}{2} \left( e^{-\varepsilon|\partial_\sigma|} \frac{\partial}{\partial x_+(\sigma, \varepsilon)} \right) = \frac{-i}{2} \left( e^{-\varepsilon|\partial_\sigma|} \frac{\partial}{\partial \tilde{x}(\sigma, \varepsilon)} \right) - i \frac{\delta_\varepsilon(\sigma, \pi/2)}{\delta_\varepsilon(0)} \frac{\partial}{\partial \bar{x}(\varepsilon)}. \quad (4.15)$$

Then the differentiation rules become more complicated, such as

$$e^{-\varepsilon|\partial_\sigma|} \frac{\partial \tilde{x}(\sigma', \varepsilon)}{\partial \tilde{x}(\sigma, \varepsilon)} = \delta_\varepsilon^+(\sigma, \sigma') - 2 \frac{\delta_\varepsilon(\sigma, \pi/2) \delta_\varepsilon(\sigma', \pi/2)}{\delta_\varepsilon(0)}, \quad (4.16)$$

which is consistent with vanishing at either  $\sigma = \pi/2$  or  $\sigma' = \pi/2$ .

Continuing in this way the star product is constructed just like Eq.(2.9) but with  $\partial/\partial \tilde{x}(\sigma, \varepsilon)$  appearing instead of  $\partial/\partial x(\sigma, \varepsilon)$ , so that it conforms to the separation of the midpoint implied by string joining in Eq.(3.3). Indeed, such a star product is guaranteed not to touch the midpoint. This is because  $\bar{x}(\varepsilon)$  is independent of  $\tilde{x}(\sigma, \varepsilon)$  and therefore  $\partial \bar{x}(\varepsilon) / \partial \tilde{x}(\sigma, \varepsilon) = 0$  insures that the midpoint is unaffected by the  $\star$ .

This reformulation can certainly be carried on, as we did for quite a while during our investigation, and wasted quite a bit of time and effort. The formalism became messy, cumbersome and obscure on some issues. However, we finally noticed that the star product (2.9) does the same job for string joining whether written in terms of  $\partial/\partial x(\sigma, \varepsilon)$  or  $\partial/\partial \tilde{x}(\sigma, \varepsilon)$ . This is because, as seen from (4.15), the difference in the star product (i.e. constructed with  $\partial/\partial x(\sigma, \varepsilon)$  as compared to  $\partial/\partial \tilde{x}(\sigma, \varepsilon)$ ) comes from the second term on the right hand side in the following equation

$$\frac{\partial}{\partial x(\sigma, \varepsilon)} \cdot \frac{\partial}{\partial p(\sigma)} = \frac{\partial}{\partial \tilde{x}(\sigma, \varepsilon)} \cdot \frac{\partial}{\partial p(\sigma)} + 2 \frac{\delta(\sigma, \pi/2)}{\delta_\varepsilon(0)} \frac{\partial}{\partial \bar{x}(\varepsilon)} \cdot \frac{\partial}{\partial p(\sigma)} \quad (4.17)$$

where the nonregulated delta appears in the numerator of the second term on the right hand side because  $e^{-\varepsilon|\partial\sigma|}$  has been removed from (4.15). However the extra piece drops out under the integral  $\int d\sigma$  in the star product,  $\delta(\sigma, \pi/2)(\partial/\partial p(\sigma)) \rightarrow 0$ , since formally  $\partial/\partial p(\pi/2) = 0$ . This shows that the string-joining  $\star$  in (2.9) could avoid the midpoint even though  $\partial/\partial x(\sigma, \varepsilon)$  in Eq.(4.17) appears to include it. We are careful to say that this argument is formal because there are delicate circumstances in which there is a midpoint contribution from the star product as noted in Eqs.(3.38-3.39). However, this is a desirable behavior of the star product in such circumstances, hence we concluded that there is no need to separate the midpoint from the rest of  $x(\sigma, \varepsilon)$ . The  $\star$  formalism in terms of the full  $x(\sigma, \varepsilon)$  greatly simplifies and becomes much easier for computations while providing new insights as will be seen in what follows.

## V. REPRESENTATIONS OF CFT OPERATORS IN MSFT

A key ingredient in the construction of SFT is the BRST operator  $\hat{Q}_B$  that appears in the quadratic kinetic term. Constructing the representation of the BRST operator in the convenient space  $(x(\sigma, \varepsilon), p(\sigma))$  is the remaining task to construct the MSFT action.

The BRST operator  $\hat{Q}_B$  can be associated to any exact conformal field theory (CFT) with any set of background fields that satisfy the exact CFT conditions. To proceed with our formulation we first define the basic (unregulated) canonical conjugates  $\hat{X}^M(\sigma), \hat{P}_M(\sigma)$  both for the string and ghost degrees of freedom from the Lagrangian for the CFT. Recall that  $X^M$  has contravariant indices and  $P_M(\sigma, \tau) = \partial S_{CFT}/\partial(\partial_\tau X^M(\sigma, \tau))$  has covariant indices; there is no metric involved in lowering the index for  $P_M$ , it has a contravariant  $M$  index for any set of background fields in the CFT. Next consider for this CFT the corresponding stress tensor  $T_{\pm\pm}(\sigma)$ , BRST current  $J_{\pm B}(\sigma)$  and BRST operator  $Q_B$ , and if desired any vertex operator, but written in terms of these canonical operators. Furthermore, perform normal ordering and insert the regulator  $\varepsilon$  so that  $\hat{T}_{\pm\pm}(\sigma, \varepsilon), \hat{J}_{\pm B}(\sigma, \varepsilon), \hat{Q}_B(\varepsilon)$  are well defined as quantum operators. In particular, insure that  $\left(\hat{Q}_B(\varepsilon)\right)^2 = 0$  as an operator in CFT when  $\varepsilon \rightarrow 0$ . As outlined in the previous section, the regulator  $\varepsilon$  is basically equivalent to the regulator implied in operator products in a CFT; with our prescription, the regulator is built in so one can proceed to computations algebraically, using only the properties of the operators, without any further reference to the CFT. We will illustrate this with an example

below.

### A. Map from CFT to MSFT and operator products

Once the steps above are performed for the CFT by using standard CFT methods, the next step is to compute the representation of these operators, in particular  $\hat{Q}_B(\varepsilon)$ , on the string field in our basis  $A(x^M(\cdot, \varepsilon), p_M(\cdot))$ . With our setup this step is straightforward because all we need to do is replace every operator  $\hat{X}^M(\sigma), \hat{P}_M(\sigma)$  that appears in the CFT operators by their representations given in Eqs.(3.23,3.24) as differential operators. But an equivalent and a much more elegant representation is the corresponding Moyal  $\star$  representation in Eqs.(3.36,3.37). Hence a CFT operator of the form  $\hat{O}(\hat{X}^M(\sigma), \hat{P}_M(\sigma))$ , where  $\hat{O}$  is some function of the canonical variables, will act on the string field  $A$  as follows

$$\hat{O}(\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)) A(x, p) = \begin{cases} O_\star(x(\sigma, \varepsilon), e^{-\varepsilon|\partial_\sigma|}p(\sigma)) \star A & \text{for } 0 \leq \sigma \leq \frac{\pi}{2}, \\ A \star O_\star(x(\sigma, \varepsilon), e^{-\varepsilon|\partial_\sigma|}p(\sigma)) (-1)^{|A||O|} & \text{for } \frac{\pi}{2} \leq \sigma \leq \pi. \end{cases} \quad (5.1)$$

Here  $(-1)^{|A||O|}$  is the sign for bose/fermi generalization. The function  $O_\star(x(\sigma, \varepsilon), e^{-\varepsilon|\partial_\sigma|}p(\sigma))$  is star multiplied on the left or right of  $A$  depending on the value of the local worldsheet parameter  $\sigma$ . The functional form of  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  is identical to the functional form of  $\hat{O}(\hat{X}, \hat{P})$ . Within the function  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  there are star products among the factors of  $x(\sigma, \varepsilon)$  or  $e^{-\varepsilon|\partial_\sigma|}p(\sigma)$  which must appear in the same order as the original quantum ordered CFT operators in  $\hat{O}(\hat{X}, \hat{P})$ . This representation is possible because the Moyal  $\star$  product is an associative product just like products of quantum operators are also associative. This map from CFT operators  $\hat{O}(\hat{X}, \hat{P})$  to their MSFT representations  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  follows directly from the map between QM to iQM and vice-versa.

If all the star products within  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  are evaluated, it reduces to a classical function of  $x(\sigma, \varepsilon), e^{-\varepsilon|\partial_\sigma|}p(\sigma)$ . The classical  $O(x, e^{-\varepsilon|\partial_\sigma|}p)$  obtained in this way is a field just like  $A(x, p)$ . Hence the representation of the CFT operator  $\hat{O}$  reduces to the Moyal  $\star$  product between two fields as follows

$$\hat{O}(\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)) A(x, p) = \begin{cases} O(x, e^{-\varepsilon|\partial_\sigma|}p) \star A(x, p) & \text{for } 0 \leq \sigma \leq \frac{\pi}{2}, \\ A(x, p) \star O(x, e^{-\varepsilon|\partial_\sigma|}p) (-1)^{|A||O|} & \text{for } \frac{\pi}{2} \leq \sigma \leq \pi, \end{cases} \quad (5.2)$$

where the functional form of  $\hat{O}(x, p)$  is closely related to  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  as just described, while  $O_\star(x, e^{-\varepsilon|\partial_\sigma|}p)$  has an identical form to the CFT operator  $\hat{O}(\hat{X}, \hat{P})$ . This transparent relationship between any CFT operator and its representation in MSFT is not only elegant, but is also useful for practical computations in string field theory in both flat and curved spaces. The reason is that now the mathematics is algebraically the same as usual quantum mechanics and we can use all we know in QM both mathematically and intuitively to perform computations in MSFT.

For clarity we provide an example of the correspondence between CFT operators and their representation as functions with star products. Consider the normal ordered  $T_{01}$  component of the matter energy-momentum tensor for the string in flat space  $\hat{T}_{01}(\hat{X}, \hat{P}) = \frac{1}{4} \left( : \pi \hat{P}(\sigma) \partial_\sigma \hat{X}(\sigma, \varepsilon) : \right)$ , where in addition to normal ordering we also introduced the regulator  $\varepsilon$ . First define the normal ordering and then apply this operator on a state in Moyal space in the case  $\sigma \leq \pi/2$

$$\begin{aligned}
& \hat{T}_{01}(\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)) A(x, p) \\
&= \frac{\pi}{4} \left( : \hat{P}(\sigma) \partial_\sigma \hat{X}_\varepsilon(\sigma, \varepsilon) : \right) A(x, p) = \frac{\pi}{4} \left( \hat{P}(\sigma) \partial_\sigma \hat{X}_\varepsilon(\sigma, \varepsilon) - \Delta'(\varepsilon) \right) A(x, p) \\
&= \frac{\pi}{4} \left( e^{-\varepsilon|\partial_\sigma|} p(\sigma) \star \partial_\sigma x(\sigma, \varepsilon) - \Delta'(\varepsilon) \right) \star A(x, p) \\
&= \frac{\pi}{4} \left( : e^{-\varepsilon|\partial_\sigma|} p(\sigma) \star \partial_\sigma x(\sigma, \varepsilon) : \right) \star A(x, p) \\
&= T_{01}^\star(x, e^{-\varepsilon|\partial_\sigma|} p) \star A(x, p)
\end{aligned} \tag{5.3}$$

where we denoted the normal ordering constant,  $\Delta'(\varepsilon) = \lim_{\sigma' \rightarrow \sigma} \partial_{\sigma'} \langle \hat{P}(\sigma) \hat{X}(\sigma', \varepsilon) \rangle$ , which is zero in this flat spacetime example. The  $\star$  product within the operator  $T_{01}^\star(x, e^{-\varepsilon|\partial_\sigma|} p) = \frac{\pi}{4} : e^{-\varepsilon|\partial_\sigma|} p(\sigma) \star \partial_\sigma x(\sigma, \varepsilon) :$  can be evaluated to finally construct the corresponding classical field  $T_{01}(x, p)$ , although this step may not be convenient to perform in some cases. For example for  $\hat{Q}_B$  it is more transparent and easier to perform computations with  $Q_B^\star$  than with the corresponding classical function  $Q_B$ .

It is worth to note that the procedure in Eq. (5.1) depends only on the canonical structure, therefore the expressions are valid for any general CFT, including any set of background fields.

The transparent relationship of the present MSFT formalism to conformal field theory is now apparent. For example, operator products in MSFT are computed with the same procedure in CFT by simply replacing products of first quantized operators in CFT by their



star product counterparts in MSFT applied on  $A(x, p)$ , such as

$$\begin{aligned} & \hat{O}^1 \left( \hat{X}(\sigma_1, \varepsilon), \hat{P}(\sigma_1) \right) \hat{O}^2 \left( \hat{X}(\sigma_2, \varepsilon), \hat{P}(\sigma_2) \right) A(x, p) \\ &= O_\star^1 \left( x(\sigma_1, \varepsilon), e^{-\varepsilon|\partial_{\sigma_1}|} p(\sigma_1) \right) \star O_\star^2 \left( x(\sigma_2, \varepsilon), e^{-\varepsilon|\partial_{\sigma_2}|} p(\sigma_2) \right) \star A(x, p), \end{aligned} \quad (5.4)$$

where we assumed both  $\sigma_1$  and  $\sigma_2$  are smaller than  $\pi/2$ . If one of them is larger than  $\pi/2$  it would appear on the right side of  $A$ . The QM to iQM map we have constructed above shows that computations of operator products in CFT,  $\hat{O}^i \hat{O}^j = c^{ijk} \hat{O}^k$ , would be reproduced one to one in MSFT by using the Moyal  $\star$  star product and yield the same operator product coefficients  $c^{ijk}$  in

$$\hat{O}_\star^i \star \hat{O}_\star^j = c^{ijk} \hat{O}_\star^k. \quad (5.5)$$

Hence we can take over all such results that are already computed in CFT and directly use them in MSFT without any further effort.

In particular, consider the stress tensor  $T_{\pm\pm}(\sigma)$  for any CFT. The CFT operator products  $\hat{T}_{\pm\pm}(\sigma_1) \hat{T}_{\pm\pm}(\sigma_2)$  are directly reproduced one to one by using the star products in MSFT  $T_{\pm\pm}^\star \left( x(\sigma_1, \varepsilon), e^{-\varepsilon|\partial_{\sigma_1}|} p(\sigma_1) \right) \star T_{\pm\pm}^\star \left( x(\sigma_2, \varepsilon), e^{-\varepsilon|\partial_{\sigma_2}|} p(\sigma_2) \right)$  when  $\sigma_1, \sigma_2$  are both on either side of the midpoint, and including the midpoint.

## B. Ghosts

Up to now we have treated the ghosts in a unified notation as the fermionic part of the  $\text{OSp}(d|2)$  vectors. In this section we are going to give the explicit connection to the  $B_{\pm\pm}(\sigma, \tau), C^\pm(\sigma, \tau)$  ghosts of conformal field theory. As usual we use capital letters  $B, C$  to denote the CFT quantities and reserve low case letters  $b, c$  for MSFT labels. Since we have the same set of ghosts for any set of conformal background fields in the matter sector of any CFT, we can consider the ghost space as being always in a flat background. Recall the mode expansion

$$\hat{B}_{\pm\pm}(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \hat{b}_n e^{-in(\tau \pm \sigma)}, \quad \hat{C}^\pm(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{-in(\tau \pm \sigma)}. \quad (5.6)$$

Let  $\hat{B}_{\pm\pm}(\sigma), \hat{C}^\pm(\sigma)$  denote the full string first quantized operators at  $\tau = 0$ . Hence  $\hat{B}_{\pm\pm}(\sigma) = \hat{B}(\pm\sigma)$  and  $\hat{C}^\pm(\sigma) = \hat{C}(\pm\sigma)$ , so we have just two operators  $\hat{B}(\sigma), \hat{C}(\sigma)$  that are each other's canonical conjugates according to the first quantization of the string. After expanding  $e^{\mp in\sigma} = \cos n\sigma \mp i \sin n\sigma$ , we define position-momentum operators  $X^{b,c}(\sigma)$ ,

$P_{b,c}(\sigma)$  as the parts of  $\hat{B}, \hat{C}$  associated with the *cosine* or *sine* series as follows

$$\hat{B}(\pm\sigma) = \left(-i\hat{X}^b(\sigma) \pm \pi\partial_\sigma\hat{P}_c(\sigma)\right), \quad \hat{C}(\pm\sigma) = \left(\pi\hat{P}_b(\sigma) \mp i|\partial_\sigma|^{-2}\partial_\sigma\hat{X}^c(\sigma)\right). \quad (5.7)$$

The  $\hat{X}^{c,b}(\sigma), \hat{P}_{c,b}(\sigma)$  are *cosine* series, just like the matter sector  $(X^\mu(\sigma), P_\mu(\sigma))$  when the latter are in flat space,

$$\hat{X}^{b,c}(\sigma) = \hat{X}_0^{b,c} + \sqrt{2} \sum_{n \geq 1} \hat{X}_n^{b,c} \cos n\sigma, \quad \pi\hat{P}_{b,c}(\sigma) = \hat{P}_{(b,c)0} + \sqrt{2} \sum_{n \geq 1} \hat{P}_{(b,c)n} \cos n\sigma, \quad (5.8)$$

but the combinations that appear in  $\hat{B}(\sigma), \hat{C}(\sigma)$ , namely  $|\partial_\sigma|^{-2}\partial_\sigma\hat{X}^c(\sigma)$  or  $\pi\partial_\sigma\hat{P}_c(\sigma)$ , are *sine* series since  $|\partial_\sigma|^k \partial_\sigma \cos n\sigma = -n^{k+1} \sin n\sigma$  for any  $k$ . The correspondence for the modes is (recall  $X^{b,c}$  are antihermitian while  $P_{b,c}$  are hermitian as explained following Eq.(2.8) )

$$\hat{X}_0^b = i\hat{b}_0, \quad \hat{X}_{n \geq 1}^b = \frac{i}{\sqrt{2}} (\hat{b}_n + \hat{b}_{-n}), \quad \hat{X}_{n \geq 1}^c = -\frac{n}{\sqrt{2}} (\hat{c}_n - \hat{c}_{-n}), \quad (5.9)$$

$$\hat{P}_{b0} = \hat{c}_0, \quad \hat{P}_{b,n \geq 1} = \frac{1}{\sqrt{2}} (\hat{c}_n + \hat{c}_{-n}), \quad \hat{P}_{c,n \geq 1} = \frac{i}{\sqrt{2}n} (\hat{b}_n - \hat{b}_{-n}), \quad (5.10)$$

and their inverse is

$$\hat{b}_0 = -i\hat{X}_0^b, \quad \hat{b}_{n \geq 1} = \frac{i}{\sqrt{2}} (-\hat{X}_{n \geq 1}^b - n\hat{P}_{c,n \geq 1}), \quad \hat{b}_{(-n \leq 1)} = \frac{i}{\sqrt{2}} (-\hat{X}_{n \geq 1}^b + n\hat{P}_{c,n \geq 1}), \quad (5.11)$$

$$\hat{c}_0 = \hat{P}_{b0}, \quad \hat{c}_{n \geq 1} = \frac{1}{\sqrt{2}} \left(-\frac{1}{n}\hat{X}_{n \geq 1}^c + \hat{P}_{b,n \geq 1}\right), \quad \hat{c}_{(-n \leq 1)} = \frac{1}{\sqrt{2}} \left(\frac{1}{n}\hat{X}_{n \geq 1}^c + \hat{P}_{b,n \geq 1}\right). \quad (5.12)$$

The remaining zero modes  $(\hat{X}_0^c, \hat{P}_{0c})$  drop out in these expressions for  $\hat{B}(\sigma), \hat{C}(\sigma)$  because  $|\partial_\sigma|^k \partial_\sigma$  applied on a constant is zero. So both  $(\hat{X}_0^c, \hat{P}_{0c})$  may be taken as zero or they may be treated as additional non-vanishing zero modes in intermediate stages of the formalism. In any case they drop out in the relevant structures of the MSFT dynamics.

We now introduce the regulated operators for ghosts,  $\hat{X}^{c,b}(\sigma, \varepsilon), \hat{P}_{c,b}(\sigma)$ , namely  $\hat{X}^{c,b}(\sigma, \varepsilon) \equiv e^{-\varepsilon|\partial_\sigma|}\hat{X}^{c,b}(\sigma)$ , while  $\hat{P}_{c,b}(\sigma)$  remain unregulated, in parallel to the matter sector. Since the  $\hat{X}^{c,b}(\sigma, \varepsilon), \hat{P}_{c,b}(\sigma)$  satisfy  $(nn)$  boundary conditions, their QM anticommutation rules are  $(m = c, b)$

$$\left\{\hat{X}^m(\sigma, \varepsilon), \hat{P}_{m'}(\sigma')\right\} = i\delta_{m'}^m \delta_\varepsilon^{nn}(\sigma, \sigma'). \quad (5.13)$$

Note that  $i$  appears on the right hand side of Eq.(5.13) in accord with the comments following Eq.(2.8). The corresponding anticommutation rules for the modes  $\hat{X}_n^{(b,c)}, \hat{P}_{(b,c)n}$  are consistent with the anticommutation rules for the ghost operators  $\hat{B}(\sigma), \hat{C}(\sigma)$  or the ghost modes  $\left\{\hat{b}_{\pm n}, \hat{c}_{\mp n'}\right\} = \delta_{nn'}$ .

Just like the matter sector, the ghosts  $X^m(\sigma, \varepsilon), P_m(\sigma)$  are split into their even and odd parts  $x_{\pm}^m(\sigma, \varepsilon), p_{\pm m}(\sigma)$  as in Eqs.(2.10,2.11) and then treated in a unified way with the bosons as part of the  $\text{OSp}(d|2)$  supervectors as we did in all previous sections. Then the string field is labeled as  $A(x_+^M(\sigma, \varepsilon), p_{-M}(\sigma))$  including the eigenvalues of the simultaneous ghost observables  $(\hat{x}_+^b, \hat{p}_{-b}, \hat{x}_+^c, \hat{p}_{-c})$ . Next we compute the representation of the full ghost operators on the string field which may be labeled by the eigenvalues as  $A(x_+^{b,c}, p_{-b,c})$  or equivalently as  $A(b, c)$  where  $(b, c)$  refer to Eqs.(5.18-5.20) below. This corresponds to specializing Eqs.(3.23,3.24), Eqs.(3.32,3.33) and Eqs.(3.36,3.37) to the case  $M = m = (b, c)$ . From these it is useful to extract the  $\star$  representation of the regulated full string ghost operators

$$\hat{B}(\pm\sigma, \varepsilon) = \left( -ie^{-\varepsilon|\partial_\sigma|} \hat{X}^b(\sigma, \varepsilon) \pm \pi \partial_\sigma \hat{P}_c(\sigma) \right), \quad (5.14)$$

$$\hat{C}(\pm\sigma, \varepsilon) = \left( \pi \hat{P}_b(\sigma) \mp ie^{-\varepsilon|\partial_\sigma|} |\partial_\sigma|^{-2} \partial_\sigma \hat{X}^c(\sigma, \varepsilon) \right), \quad (5.15)$$

as follows

$$\hat{B}(+\sigma, \varepsilon) A(b, c) = \begin{cases} b(\sigma, \varepsilon) \star A(b, c), & \text{if } 0 \leq \sigma \leq \pi/2 \\ A(b, c) \star b(\sigma, \varepsilon) (-1)^A, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}, \quad (5.16)$$

and

$$\hat{C}(+\sigma, \varepsilon) A(b, c) = \begin{cases} c(\sigma, \varepsilon) \star A(b, c), & \text{if } 0 \leq \sigma \leq \pi/2 \\ A(b, c) \star c(\sigma, \varepsilon) (-1)^A, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}. \quad (5.17)$$

The low case  $b(\sigma, \varepsilon), c(\sigma, \varepsilon)$  correspond to the following combinations of  $(x_+^{b,c}, p_{-b,c})$

$$\begin{aligned} b(\sigma, \varepsilon) &= e^{-\varepsilon|\partial_\sigma|} (-ix_+^b(\sigma, \varepsilon) + \pi \partial_\sigma p_{-c}(\sigma)), \\ c(\sigma, \varepsilon) &= e^{-\varepsilon|\partial_\sigma|} (\pi p_{-b}(\sigma) - i|\partial_\sigma|^{-2} \partial_\sigma x_+^c(\sigma, \varepsilon)). \end{aligned} \quad (5.18)$$

When written in terms of the modes, this gives the regulated  $b(\sigma, \varepsilon), c(\sigma, \varepsilon)$  in terms of the unregulated odd/even modes of the ghosts in Eq.(5.6)

$$b(\sigma, \varepsilon) = b_0 + \sum_{o \geq 1} (b_e + b_{-e}) e^{-2\varepsilon e} \cos e\sigma - i \sum_{e \geq 2} (b_o - b_{-o}) e^{-\varepsilon o} \sin o\sigma, \quad (5.19)$$

$$c(\sigma, \varepsilon) = \sum_{e \geq 2} (c_o + c_{-o}) e^{-\varepsilon o} \cos o\sigma - i \sum_{o \geq 1} (c_e - c_{-e}) e^{-2\varepsilon e} \sin e\sigma, \quad (5.20)$$

where  $e = 2, 4, 6, \dots$  are positive even integers and  $o = 1, 3, 5, \dots$  are positive odd integers. Clearly  $(b(\sigma, \varepsilon), c(\sigma, \varepsilon))$  is only half of the ghost phase space in (5.6). Note that  $c(\sigma, \varepsilon)$  has

no zero mode as mentioned after Eq.(5.10). Furthermore,  $b$  is even while  $c$  is odd under reflections from the midpoint

$$b(\sigma, \varepsilon) = b(\pi - \sigma, \varepsilon), \quad c(\sigma, \varepsilon) = -c(\pi - \sigma, \varepsilon). \quad (5.21)$$

The  $(b(\sigma, \varepsilon), c(\sigma, \varepsilon))$  given above are constructed from the eigenvalues of the ghost operators,  $\hat{b}_n$  and  $\hat{c}_n$ , in such combinations that clearly anticommute with each other under the standard QM rules. By contrast, under the string-joining  $\star$  product in (2.9)  $(b(\sigma, \varepsilon), c(\sigma, \varepsilon))$  do not commute with each other in the induced iQM. Using Eq.(3.25), and (5.18), we compute that they satisfy the following iQM anticommutation rule

$$\{b(\sigma_1, \varepsilon), c(\sigma_2, \varepsilon)\}_\star = 2\pi e^{-\varepsilon} |\partial_{\sigma_1}| e^{-\varepsilon} |\partial_{\sigma_2}| \hat{\delta}^{nn+dd}(\sigma_1, \sigma_2), \quad (5.22)$$

where

$$\hat{\delta}^{nn+dd}(\sigma_1, \sigma_2) = \frac{1}{2} \left( \delta^{+nn}(\sigma_1, \sigma_2) \text{sign}\left(\frac{\pi}{2} - \sigma_2\right) + \text{sign}\left(\frac{\pi}{2} - \sigma_1\right) \delta^{-dd}(\sigma_1, \sigma_2) \right). \quad (5.23)$$

where  $\delta^{+nn}(\sigma_1, \sigma_2)$ ,  $\delta^{-dd}(\sigma_1, \sigma_2)$  are given in (4.6-4.9) and figures Fig. 1,2. The *sign* functions that appear in  $\hat{\delta}^{nn+dd}(\sigma_1, \sigma_2)$  are essential. This expression obeys similar relations to Eqs.(3.26-3.27). From the midpoint properties of  $\hat{\delta}$ , namely  $\hat{\delta}(\pi/2, \sigma_2) = 0 = \hat{\delta}(\sigma_1, \pi/2)$ , we see that the midpoint ghost degrees of freedom act trivially under the string joining  $\star$  product, just as desired.

In practical computations sometimes it is useful to use the star product directly in terms of  $b, c$  as in Eq.(5.22) or sometimes revert back to  $x_+^{b,c}, p_{-b,c}$  through Eq.(5.18) and write everything in terms of  $x^M(\sigma, \varepsilon), p_M(\sigma)$  to take advantage of the  $\text{OSp}(d|2)$  symmetry of the  $\star$  product. The  $(b, c)$  basis is useful for constructing the representation of the BRST operator as in the next section.

### C. Stress tensor, BRST current and BRST operator

In this section we discuss the BRST charge and the associated building blocks, stress tensor and BRST current, in the MSFT formalism. The plan is to first define the regulated first quantized operators in QM and then construct their iQM representations in terms of only the string-joining Moyal star product

### 1. Regulated QM operators

The BRST operator in QM is defined as an integral over the left and right moving BRST currents for the full string [27]

$$\hat{Q}_B(\varepsilon) = \frac{1}{\pi} \int_0^\pi \hat{J}_B(\sigma, \varepsilon) d\sigma, \text{ with } \hat{J}_B(\sigma, \varepsilon) \equiv \frac{1}{2} \left( \hat{J}_+^B(\sigma, \varepsilon) + \hat{J}_-^B(\sigma, \varepsilon) \right), \quad (5.24)$$

where we also introduce the regulator  $\varepsilon$  as shown below. In the QM operator formalism the regulated and normal ordered current is defined as

$$\hat{J}_\pm^B(\sigma, \varepsilon) =: \left( 2\hat{C}(\pm\sigma, \varepsilon) \left( \hat{T}_{\pm\pm}^m(\sigma, \varepsilon) + \frac{1}{2}\hat{T}_{\pm\pm}^{gh}(\sigma, \varepsilon) \right) + a\hat{C}(\pm\sigma, \varepsilon) \right. \\ \left. + \frac{3}{2}(-i\partial_{\pm\sigma})(-i\partial_{\pm\sigma} + 1)\hat{C}(\pm\sigma, \varepsilon) \right) :, \quad a = -1 \quad (5.25)$$

When  $\varepsilon \rightarrow 0$  this agrees with standard definitions, e.g. see [27], [28]. The total derivative term in the second line drops out in the computation of the integral BRST operator, but is needed to insure that  $\hat{J}_\pm^B$  is a conformal tensor. The constant coefficient  $a$  arises from normal ordering, and is fixed to  $a = -1$  by requiring the BRST operator to satisfy  $\hat{Q}_B^2 = 0$ .

To provide an example of the consistent regularization, we take the case of the flat space CFT with trivial background fields, where the regularized operator for the matter energy-momentum tensor  $\hat{T}_{\pm\pm}^m(\sigma, \varepsilon)$  takes the form

$$\hat{T}_{\pm\pm}^m(\sigma, \varepsilon) =: \frac{1}{4} \left[ \pi \hat{P}^\mu(\sigma) \pm e^{-\varepsilon|\partial_\sigma|} \partial_\sigma \hat{X}^\mu(\sigma, \varepsilon) \right]^2 : \quad (5.26)$$

with the regulator  $\varepsilon$  included. There is also an identical form for the regulated ghost stress tensor, which is the same for all CFTs, as given below in Eq.(5.33). In these expressions we could have written  $e^{-\varepsilon|\partial_\sigma|} \hat{X}(\sigma, \varepsilon) = \hat{X}(\sigma, 2\varepsilon)$ , but we should keep it as given in (5.26) because we are committed to the notation that the independent degrees of freedom are designated as  $\hat{X}(\sigma, \varepsilon)$ . The reader familiar with string theory can verify that the expressions above revert to the familiar unregulated expressions in the  $\varepsilon \rightarrow 0$  limit [27].

Next we introduce the regulated ghost stress tensor  $\hat{T}_{\pm\pm}^{gh}(\sigma, \varepsilon)$  in terms of the regulated  $\hat{C}(\pm\sigma, \varepsilon)$ ,  $\hat{B}(\pm\sigma, \varepsilon)$  operators of Eq.(5.14), and compute it as follows

$$\hat{T}_{\pm\pm}^{gh} =: i \left( \partial_{\pm\sigma} \hat{C}(\pm\sigma, \varepsilon) \hat{B}(\pm\sigma, \varepsilon) + \frac{1}{2} \hat{C}(\pm\sigma, \varepsilon) \partial_{\pm\sigma} \hat{B}(\pm\sigma, \varepsilon) \right) : \quad (5.27)$$

$$=: \frac{i}{2} \left( \partial_{\pm\sigma} \hat{C}(\pm\sigma, \varepsilon) \hat{B}(\pm\sigma, \varepsilon) + \partial_{\pm\sigma} \left( \hat{C}(\pm\sigma, \varepsilon) \hat{B}(\pm\sigma, \varepsilon) \right) \right) : \\ = \frac{i}{2} : \left( \begin{array}{c} e^{-\varepsilon|\partial_\sigma|} \hat{X}^c(\sigma, \varepsilon) \\ \mp \pi i \partial_\sigma \hat{P}_b(\sigma) \end{array} \right) \left( \begin{array}{c} e^{-\varepsilon|\partial_\sigma|} \hat{X}^b(\sigma, \varepsilon) \\ \pm i \pi \partial_\sigma \hat{P}_c(\sigma) \end{array} \right) : - \frac{1}{2} i \partial_{\pm\sigma} J_\pm^{gh}(\sigma, \varepsilon). \quad (5.28)$$

where,  $J_{\pm}^{gh}(\sigma, \varepsilon) = \hat{C}(\pm\sigma, \varepsilon) \hat{B}(\pm\sigma, \varepsilon)$ , is the ghost number current density<sup>1</sup>.  $\hat{T}_{\pm\pm}^{gh}(\sigma, \varepsilon)$  can be written in a more appealing Sp(2)-invariant form by raising and lowering indices in the ghost sector,  $\hat{P}^m = -i\epsilon^{mm'} \hat{P}_{m'}$  etc. (namely  $P^b = -iP_c$  and  $P^c = iP_b$ ), using the antisymmetric Sp(2) metric  $\eta_{mm'} \equiv i\epsilon_{mm'}$ , which satisfies

$$(-i\epsilon^{mm'}) (i\epsilon_{m'n}) = \delta_n^m : \epsilon^{bc} = -\epsilon^{cb} = -\epsilon_{bc} = \epsilon_{cb} = 1. \quad (5.32)$$

Then we can write the manifestly Sp(2) invariant expression for  $\hat{T}_{\pm\pm}^{gh}$

$$\hat{T}_{\pm\pm}^{gh} = \frac{1}{4} (i\epsilon_{mm'}) : \left( \begin{array}{c} \pi \partial_{\sigma} \hat{P}^m(\sigma) \\ \mp e^{-\varepsilon|\partial_{\sigma}|} \hat{X}^m(\sigma, \varepsilon) \end{array} \right) \left( \begin{array}{c} \pi \partial_{\sigma} \hat{P}^{m'}(\sigma) \\ \mp e^{-\varepsilon|\partial_{\sigma}|} \hat{X}^{m'}(\sigma, \varepsilon) \end{array} \right) : - \frac{1}{2} i \partial_{\pm\sigma} J_{\pm}^{gh}, \quad (5.33)$$

It is now interesting to note the similarities and differences between the ghost stress tensor and the matter stress tensor in flat space given in Eq.(5.26). Setting aside the extra total derivative term  $\frac{1}{2} \partial_{\pm\sigma} J_{\pm}^{gh}$ , the structure of  $\hat{T}_{\pm\pm}^{gh}$  is similar to  $\hat{T}_{\pm\pm}^{matter}$  in flat space except for the following differences: the metric in flat space is the Minkowski metric  $\eta_{\mu\nu}$  whereas the metric in ghost space is the Sp(2) metric  $\eta_{mn} = i\epsilon_{mn}$ ; furthermore the  $\partial_{\sigma}$  derivative structure is different; however had the derivative structure been the same as the bosonic sector then there would have been an OSp(d|2) symmetry in the kinetic energy operator (see however section (V F) below for an improved supersymmetric basis). Consider the zeroth Virasoro operator  $L_0 = \frac{1}{\pi} \int_0^{\pi} d\sigma \sum_{\pm} \left( T_{\pm\pm}^{matter} + \hat{T}_{\pm\pm}^{gh} \right)$  which is the kinetic energy operator in the Siegel gauge as seen below. Doing integration by parts in the  $\sigma$  integral, and recalling  $|\partial_{\sigma}| = \sqrt{-\partial_{\sigma}^2}$ , then  $\hat{L}_0$  for the flat CFT case takes the form

$$\hat{L}_0^{(flat)} = \frac{1}{\pi} \int_0^{\pi} d\sigma : \left( \begin{array}{c} \frac{1}{2} \eta_{\mu\nu} \left( \pi^2 \hat{P}^{\mu} \hat{P}^{\nu} + \hat{X}^{\mu} |\partial_{\sigma}|^2 \hat{X}^{\nu} \right) \\ + \frac{1}{2} i \epsilon_{mn} \left( \pi^2 \hat{P}^m |\partial_{\sigma}|^2 \hat{P}^n + \hat{X}^m \hat{X}^n \right) \end{array} \right) :, \quad (5.34)$$

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<sup>1</sup> The regulated ghost number operator is (after dropping total derivatives in the integrations below)

$$\begin{aligned} \hat{N}_{gh} &= \frac{1}{2\pi} \int_0^{\pi} d\sigma \sum_{\pm} : \hat{C}(\pm\sigma, \varepsilon) \hat{B}(\pm\sigma, \varepsilon) : \\ &= \frac{1}{\pi} \int_0^{\pi} d\sigma : \left( i e^{-\varepsilon|\partial_{\sigma}|} \hat{X}^b(\sigma, \varepsilon) \pi \hat{P}_b(\sigma) - i e^{-\varepsilon|\partial_{\sigma}|} \hat{X}^c(\sigma, \varepsilon) \pi \hat{P}_c(\sigma) \right) : \end{aligned} \quad (5.29)$$

$$= g_0 + i \left( \hat{X}_0^b \hat{P}_{0b} - \hat{X}_0^c \hat{P}_{0c} + \dots \right) \quad (5.30)$$

$$= g_0 - x_0^b \frac{\partial}{\partial x_0^b} + x_0^c \frac{\partial}{\partial x_0^c} + \dots, \quad (5.31)$$

where  $g_0$  is a charge associated with the string state. For the string field  $A$  that appears in the action we have  $g_0 = 2$ . See also below.

This shows more clearly how the  $\text{OSp}(d|2)$  symmetry of the star product in the cubic term of the action is broken in the quadratic term down to  $\text{OSp}(d) \times \text{Sp}(2)$ .

For the example of flat CFT the regulated BRST current takes the form

$$\hat{j}_B^{(flat)}(\sigma, \varepsilon) =: \left( \pi \hat{P}_b \left[ \left( \pi^2 \hat{P}^\mu \hat{P}_\mu + \hat{X}'^\mu \hat{X}'_\mu \right) + \left( i\pi^2 \hat{P}'_b \hat{P}'_c + i\hat{X}^c \hat{X}^b \right) + 1 \right] + i\pi \partial_\sigma^{-1} \hat{X}^c \left[ \hat{X}'^\mu \hat{P}_\mu + \hat{P}'_c \hat{X}^c + \hat{P}'_b \hat{X}^b \right] + \partial_\sigma(\dots) \right) :, \quad (5.35)$$

where all  $\hat{X}^M$  should be replaced by their regulated form  $e^{-\varepsilon|\partial_\sigma|} \hat{X}^M(\sigma, \varepsilon)$  while all  $\hat{P}_M$  remain unregulated  $\hat{P}_M(\sigma)$ . The BRST operator is the integral of this current,  $\hat{Q}_B^{(flat)} = \int_0^\sigma d\sigma \hat{j}_B^{(flat)}(\sigma, \varepsilon)$ . As is well known, the vanishing  $\hat{Q}_B^2 = 0$  requires  $d = 26$ .

## 2. iQM Representation of Regulated QM Operators

Having defined the regulated version of the first quantized operators, next we write the iQM representations of all these QM operators when they act on the string field  $A(x, p)$  by following the prescription given in the previous section in Eq.(5.1). In particular, we are interested in all values of  $\sigma$  below or above the midpoint. Hence, for the BRST current we have (with step functions as in Eqs.(3.32,3.33))

$$\hat{j}_B(\sigma, \varepsilon) A(x, p) = \left[ \begin{array}{l} \theta\left(\frac{\pi}{2} - \sigma\right) j_\star(\sigma, \varepsilon) \star A(x, p) \\ + \theta\left(\sigma - \frac{\pi}{2}\right) A(x, p) \star j_\star(\sigma, \varepsilon) (-1)^A \end{array} \right] \quad (5.36)$$

The iQM image of the BRST current  $j_{\pm\star}(\sigma, \varepsilon)$  is expressed in terms of the half phase space in both the matter and ghost sectors. In the previous sections these were lumped together as supervectors  $x^M(\sigma, \varepsilon), p_M(\sigma)$  where the  $x^{b,c}(\sigma, \varepsilon)$  and  $p_{b,c}(\sigma)$  components referred to the ghosts. As explained in the next section, a combination of  $x^{b,c}, p_{b,c}$  is written in terms of  $b(\sigma, \varepsilon), c(\sigma, \varepsilon)$  that form the half phase space more directly related to the operators  $\hat{B}, \hat{C}$  as given in Eqs.(5.16,5.17) which are the equivalent of Eq.(3.32,3.33) in just the ghost sector. It is suggestive to write the Moyal basis expression for  $J_{\pm\star}(\sigma, \varepsilon)$  in terms of  $b(\sigma, \varepsilon), c(\sigma, \varepsilon)$  rather than  $x^{b,c}(\sigma, \varepsilon)$  and  $p_{b,c}(\sigma)$  as follows so that the iQM parallel to the operator QM version in (5.25) is evident

$$j_\star(\sigma, \varepsilon) = \sum_{\pm} : \left( 2c(\pm\sigma, \varepsilon) \left( T_{\pm\pm\star}^m(\sigma, \varepsilon) + \frac{1}{2} T_{\pm\pm\star}^{gh}(\sigma, \varepsilon) \right) + ac(\pm\sigma, \varepsilon) \right. \\ \left. + \frac{3}{2} (-i\partial_{\pm\sigma}) ((-i\partial_{\pm\sigma}) + 1) c(\pm\sigma, \varepsilon) \right) : . \quad (5.37)$$

Taking the integral of both sides of Eq.(5.36), we obtain the representation of the QM BRST charge in the MSFT formalism

$$\begin{aligned}\hat{Q}_B A(x, p) &= \left( \frac{1}{2\pi} \int_0^{\pi/2} d\sigma j_\star(\sigma, \varepsilon) \right) \star A(x, p) + (-1)^A A(x, p) \star \left( \frac{1}{2\pi} \int_{\pi/2}^\pi d\sigma j_\star(\sigma, \varepsilon) \right) \\ &= Q(x, p) \star A(x, p) - (-1)^A A(x, p) \star Q(x, p).\end{aligned}\quad (5.38)$$

The relative minus sign in the second term on the second line arises because  $j_\star(\sigma, \varepsilon)$  is antisymmetric relative to the midpoint  $j_\star(\pi - \sigma, \varepsilon) = -j_\star(\sigma, \varepsilon)$ , leading to

$$Q(x, p) \equiv \frac{1}{2\pi} \int_0^{\pi/2} d\sigma j_\star(\sigma, \varepsilon) = -\frac{1}{2\pi} \int_{\pi/2}^\pi d\sigma j_\star(\sigma, \varepsilon) = \frac{1}{2\pi} \int_0^\pi d\sigma \text{sign}\left(\frac{\pi}{2} - \sigma\right) j_\star(\sigma, \varepsilon). \quad (5.39)$$

Hence, the representation of the QM BRST operator reduces to the super-commutator (5.38) in iQM in the MSFT formalism. It must be emphasized that the integral that defines the string field  $Q(x, p)$  in Eq.(5.39) is over the half string. For the CFT with flat background  $Q(x, p)$  is given by (normal ordering is implied, and ' means  $\partial_\sigma$ )

$$Q(x, p) = \frac{1}{2\pi} \int_0^{\pi/2} d\sigma \left[ \begin{aligned} &\pi p_b (\pi^2 p^\mu p_\mu + x'^\mu x'_\mu + i\pi^2 p'_b p'_c + i x^c x^b) \\ &+ i\partial_\sigma^{-1} x^c (\pi p_\mu x'^\mu + \pi p'_c x^c + \pi p'_b x^b) \end{aligned} \right] \quad (5.40)$$

where all  $x^M$  ( $p_M$ ) are symmetric (antisymmetric) under reflections from the midpoint and appear everywhere in  $Q(x, p)$  in the regulated forms

$$x^M \rightarrow e^{-\varepsilon|\partial_\sigma|} x_+^M(\sigma, \varepsilon), \quad p_M \rightarrow e^{-\varepsilon|\partial_\sigma|} p_{-M}(\sigma). \quad (5.41)$$

For the general CFT the matter part in (5.40) is modified as follows, while the ghost sector in (5.40) is common to all CFTs so it remains unchanged. We begin with the matter stress tensor  $T^{matter}(\sigma, \varepsilon)$  for the desired CFT written in terms of phase space  $(X^\mu, P_\mu)$  by using the standard canonical procedure (replacing all velocities by momenta). Recall that in the general CFT all positions are defined with an upper index  $X^\mu$  while the lower index in  $P_\mu$  naturally follows from the CFT action  $S_{CFT}(X, \partial X)$  by using the standard canonical procedure  $P_\mu = \partial S_{CFT} / \partial(\partial_\tau X^\mu)$ . All operators  $(\hat{X}^\mu, \hat{P}_\mu)$  in  $\hat{T}^{matter}(\sigma, \varepsilon)$  are quantum ordered to insure  $\hat{T}^{matter}$  is normal ordered and has the correct quantum properties as the generator of conformal transformations on the worldsheet. After this step the operators  $(\hat{X}^\mu, \hat{P}_\mu)$  are replaced by half of the phase space  $(x^\mu, p_\mu)$  in the regulated form shown in



(5.41), in the same order as the operators but with star products  $\star$  inserted in between non-commuting factors. Then this  $\hat{T}^{matter}(\sigma, \varepsilon)$  is inserted in the BRST operator to obtain the BRST field  $Q(x, p)$  that modifies (5.40) to a general CFT.

As explained just before Eq.(5.2) by evaluating all the star products within  $Q(x, p)$  it can be expressed as a classical function  $Q(x, p)$  which is a string field just like  $A(x, p)$ . For example, for the flat CFT, evaluating all the star products in (5.37) is equivalent to forgetting all the  $\star$ 's and replacing the constant  $a$  by a shifted value that depends on the regulator  $\varepsilon$ .

In the operator formalism the normal ordered BRST charge in any CFT is quantum ordered to be nilpotent

$$\hat{Q}_B^2 = 0. \quad (5.42)$$

For the flat CFT background this condition is satisfied at the critical dimension  $d = 26$  and intercept  $a = -1$ . In the MSFT approach the nilpotency property for any CFT has the following interesting consequence by using the associativity property of the  $\star$

$$\begin{aligned} 0 &= \hat{Q}_B^2 A = \hat{Q}_B \left( Q \star A - (-1)^A A \star Q \right) \\ &= Q \star \left( Q \star A - (-1)^A A \star Q \right) - (-1)^{A+1} \left( Q \star A - (-1)^A A \star Q \right) \star Q \\ &= (Q \star Q) \star A - A \star (Q \star Q) \\ &= [(Q \star Q), A]_\star \end{aligned} \quad (5.43)$$

Since this commutator must vanish for any  $A$ , we conclude that the star product of the fields  $Q \star Q$  must be a constant, and possibly zero. To figure out what the constant is we need to compute the following star-anticommutator

$$Q \star Q = \frac{1}{2} \{Q, Q\}_\star = \frac{1}{4\pi} \int_0^{\pi/2} d\sigma_1 \int_0^{\pi/2} d\sigma_2 \{j_\star(\sigma_1, \varepsilon), j_\star(\sigma_2, \varepsilon)\}_\star. \quad (5.44)$$

This anticommutator naively would be zero for two fermions; however the star product turns them essentially into non-anticommuting operators in iQM such that the anticommutator has support only in sharply local regions where  $\sigma_1$  approaches  $\sigma_2$ . Given the map we have established between iQM and QM, the exact parallel of this iQM computation can be done in the operator QM version in any CFT. Therefore, we can simply take over the known universal results for the *local* operator products  $\left\{ \hat{J}_B(\sigma_1, \varepsilon), \hat{J}_B(\sigma_2, \varepsilon) \right\}$  for the BRST current in any CFT [e.g. [28] Eq.(4.3.10)], restrict  $\sigma_1, \sigma_2$  to only the left (or right) half of the string, and

integrate  $\sigma_1, \sigma_2$  in the range for half of the string  $[0, \pi/2]$ . The same local computation was used to prove  $\hat{Q}_B^2 = 0$  but in that case the range of the integrals is  $[0, \pi]$ . The result of the half-string integrals is the same as the full string integrals because the support comes only from sharply local regions, in our case within the region  $0 \leq \sigma_1, \sigma_2 \leq \pi/2$ . The non-trivial anticommutator has three terms [28]:  $\hat{Q}_B^2 = \int \int \sum_{i=0}^2 \hat{\alpha}_i(\sigma_1) \delta^{(i)}(\sigma_1, \sigma_2)$ , where  $\delta^{(i)}$  are the zeroth, first and second derivatives of the delta function  $i = 0, 1, 2$ . The integrals vanish for  $i = 1, 2$ , while the coefficient  $\alpha_0$  is proportional to  $(c - 26) \left( \hat{C} \partial_\sigma^2 \hat{C} \right)$  so it vanishes for any critical CFT  $c = 26$ . Therefore the field  $Q(x, p)$  is actually nilpotent provided the corresponding CFT satisfies the criticality conditions, namely the same conditions for which  $\hat{Q}_B^2 = 0$ . Hence, for the flat CFT

$$Q(x, p) \star Q(x, p) = 0, \text{ iff } c = 26, \text{ and } a = 1, \quad (5.45)$$

where  $c$  is the central charge of the CFT and  $a$  is the “intercept”. For the general CFT, provided the background fields have the correct properties at the quantum level for the theory to be a CFT, namely  $\hat{Q}_B^2 = 0$  for the operator, then the corresponding  $Q(x, p)$  constructed with the prescription above will also satisfy  $Q(x, p) \star Q(x, p) = 0$  under the string-joining Moyal  $\star$  product.

We have established that the BRST QM operator  $\hat{Q}_B$  is represented in iQM as a super-inner product with the nilpotent field  $Q(x, p)$  as given in Eq.(5.39).

#### D. The MSFT action

To construct the action for MSFT we need three ingredients: the star product, the BRST operator, and a rule for integration that has the property  $\int \hat{Q}_B A = 0$ . We already have the first two, now we define the rule for integration which is equivalent to a supertrace or a super-integral over all the phase space degrees of freedom in the induced quantum mechanics

$$\text{Str}(A) \equiv \int (Dx_+^M(\cdot, \varepsilon)) (Dp_{-M}(\cdot)) A(x, p). \quad (5.46)$$

This super-integral in phase space is indeed a supertrace of an operator  $A$  in iQM, as is well known in the Moyal product literature. So using the notions of supertrace and its cyclic properties, we immediately know that the supertrace of a supercommutator is always zero,

hence the desired property  $\int \hat{Q}_B A = 0$  is satisfied

$$\int \hat{Q}_B A = \text{Str}([Q, A]_\star) = 0. \quad (5.47)$$

To see this result directly as the property of the phase space integral, we compute  $\int \hat{Q}_B A$  by using the iQM representation (5.38) of the BRST operator in terms of the field  $Q(x, p)$  as follows

$$\int \hat{Q}_B A = \int (Dx Dp) \left( Q(x, p) \star A(x, p) - (-1)^A A(x, p) \star Q(x, p) \right). \quad (5.48)$$

Under the integral the  $\star$  can be removed when there are only two fields because each non-trivial piece of the Moyal  $\star$  (2.9) can be rewritten as a total derivative and would lead to vanishing terms at the boundaries of phase space where  $A(x, p)$  is assumed to vanish. The remaining ordinary product between the two classical string fields  $(QA - (-1)^A AQ)$  is trivially zero since the second term cancels the first term after interchanging the orders of  $A$  and  $Q$ . Hence, once again we have proven  $\int \hat{Q}_B A = 0$ , so we have chosen a good integration rule.

Now we can convert the cubic action for open string field theory proposed by Witten [1]

$$S(\Psi) = -\text{Str} \left( \frac{1}{2} \Psi \star (\hat{Q}_B \Psi) + \frac{g_0}{3} \Psi \star \Psi \star \Psi \right) \quad (5.49)$$

to our new MSFT formalism by the rules developed in the previous sections for representing QM operators. In particular, for a fermionic  $A(x, p)$  (i.e.  $(-1)^A = -1$ ) recall that the BRST operator  $\hat{Q}_B$  is represented in iQM with an anticommutator involving the field  $Q(x, p)$

$$\hat{Q}_B A = \{Q, A\}_\star = Q \star A + A \star Q. \quad (5.50)$$

The fermionic field  $Q(x, p)$  is the specific string field given in Eq.(5.39) for any CFT (or Eq.(5.40) for the flat CFT).

We digress temporarily to discuss ghost numbers of string fields before we construct the action. The ghost number operator  $\hat{N}_{gh}$  has the following representation when applied on any string field  $A(x, p)$

$$\hat{N}_{gh} A = \frac{1}{2} \int_0^\pi d\sigma \left( \begin{array}{c} \frac{2}{\pi} \hat{g} + x^c(\sigma) \frac{\partial}{\partial x^c(\sigma)} + p_b(\sigma) \frac{\partial}{\partial p_b(\sigma)} \\ -x^b(\sigma) \frac{\partial}{\partial x^b(\sigma)} - p_c(\sigma) \frac{\partial}{\partial p_c(\sigma)} \end{array} \right) A \quad (5.51)$$

$$= \left( \begin{array}{c} g_A + x_e^c \frac{\partial}{\partial x_e^c} + p_{ob} \frac{\partial}{\partial p_{ob}} \\ -x_0^b \frac{\partial}{\partial x_0^b} - x_e^b \frac{\partial}{\partial x_e^b} - p_{oc} \frac{\partial}{\partial p_{oc}} \end{array} \right) A, \quad (5.52)$$

where we used  $x_0^b = ib_0$  and there is an implied summation over the even and odd integer modes,  $e = 2, 4, 6, \dots$  and  $o = 1, 3, 5, \dots$ . The operator  $\hat{g}A = g_A A$  gives an eigenvalue  $g_A$  that corresponds to a ghost charge  $g_A$  assigned to the field  $A$ . The eigenvalue  $g_A$  may differ for various fields  $A(x, p)$ . The star products of any  $A$  with the field  $Q$ , namely the  $Q \star A$  or  $A \star Q$  that appear in (5.50), must have the total  $\hat{N}_{gh}$  increased by  $+1$  relative to  $A$  since the operator  $\hat{Q}_B$  has ghost number  $+1$ , namely  $[\hat{N}_{gh}, \hat{Q}_B] = +\hat{Q}_B$ . This implies that, for the special field  $Q(x, p)$

$$\hat{N}_{gh}Q(x, p) = (+1)Q(x, p) . \quad (5.53)$$

Taking into account the explicit form of  $Q(x, p)$  in (5.39, 5.40) and the action of the derivatives in (5.51) on this  $Q$ , we conclude that the eigenvalue of  $\hat{g}$  that appears in (5.51) when applied on  $Q$  is zero  $g_Q = 0$ .

We now consider the kinetic term of the Witten action (5.49) transcribed to our basis

$$\begin{aligned} S_{kin}(A) &= -\text{Str} \left( \frac{1}{2} A \star (\hat{Q}_B A) \right) = -\text{Str} \left( \frac{1}{2} A \star \{Q, A\}_\star \right) \\ &= -\text{Str}(A \star Q \star A) . \end{aligned} \quad (5.54)$$

In the last line we used the cyclic property of the Str to simplify the kinetic term to its final form. In Eq.(5.54) the  $\star$  product is our regulated Moyal  $\star$  product of Eq.(2.9) which has zero ghost number, and the supertrace is the phase space integral over half of the full string's classical phase space as in Eq.(5.46), which includes the ghost zero mode  $x_0^b$  or equivalently the ghost midpoint  $\bar{x}^b \equiv x^b(\pi/2)$  mode. This integration rule<sup>2</sup> has ghost number  $+1$  (because of the  $\int dx_0^b$  or  $\int d\bar{x}^b$ ), therefore the integrand, or the argument of

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<sup>2</sup> The Str or integration rule  $\int$  contains an equal number of *non-zero*  $b, c$  ghost modes  $(x_e^b, x_e^c, p_{ob}, p_{oc})$ , where  $e = 2, 4, 6, \dots$  labels even modes and  $o = 1, 3, 5, \dots$  labels odd modes, while both  $A$  and  $\int$  depend on only one ghost zero mode  $x_0^b$ . The other ghost zero mode,  $\hat{c}_0$ , is not in the half phase space: it acts on  $A(x, p)$  as a derivative  $\hat{c}_0 A(x, p) = i\partial_{b_0} A(x, p)$ . Since  $b, c$  have opposite ghost numbers, the ghost numbers of the non-zero modes cancel out in the integration rule, leaving unbalanced only the zero mode  $dx_0^b$ . Hence the total integration rule has ghost number  $+1$ , which is opposite to that of  $x_0^b$ , since Grassman integration is defined as being equivalent to the derivative,  $\int dx_0^b \times x_0^b = \partial_{x_0^b} x_0^b = 1$ . Another approach for the same result is to consider the measure of integration in the  $\sigma$  basis. which contains at each  $\sigma \neq \pi/2$  the ghost pairs  $Dx_+^b(\sigma) Dp_{-b}(\sigma)$  and similarly  $Dx_+^c(\sigma) Dp_{-c}(\sigma)$ . At each  $\sigma \neq \pi/2$  the measure of integration has ghost number 0 because  $x, p$  have opposite ghost numbers. However at  $\sigma = \pi/2$ , there is an additional unpaired  $Dx_+^b(\pi/2)$  with ghost number  $+1$  (opposite to ghost number of  $x^b$ ). This is because all momenta  $p^{b,c}(\sigma)$  vanish at  $\sigma = \pi/2$  while  $x_+^c(\pi/2)$  is not integrated as an additional independent mode since  $x^c(\sigma)$  has no zero mode as explained after Eq.(5.10). Thus the unpaired midpoint mode  $\bar{x}^b \equiv x_+^b(\pi/2)$  causes the integration rule  $\int d\bar{x}^b$  or the Str to have ghost number  $+1$ .

the Str, namely  $(A \star Q \star A)$ , must have ghost number  $-1$ . Accordingly, taking into account (5.53), the total ghost number of the field  $A$  in Eq.(5.54) must be  $-1$ . Then, for the field  $A$  that appears in the action we must have total ghost numbers assigned as follows

$$\hat{N}_{gh}A(x, p) = -A(x, p), \quad \hat{N}_{gh}(Q(x, p) \star A(x, p)) = 0. \quad (5.55)$$

This implies that in Eq.(5.51) a nontrivial  $\hat{g}A = g_A A$ , is included in the total ghost number  $\hat{N}_{gh}A = (-1)A$ . We will see that the perturbative vacuum, including zero modes of the ghosts, will require  $g_A = 2$  for  $A(x, p)$ .

Next we turn to the cubic interaction term. Given  $\hat{N}_{gh}A = -A$  and  $\hat{N}_{gh}\text{Str} = (+1)\text{Str}$ , a straightforward term of the form  $\text{Str}(A \star A \star A)$  is inconsistent with total zero ghost number for the action. Hence there has to be some midpoint insertions to achieve the correct zero total ghost number. There is a unique answer that gives the final form of the total action as follows

$$S(A) = -\text{Str} \left( A \star Q \star A + \frac{g_0}{3} A \star \partial_{\bar{b}} A \star \partial_{\bar{b}} A \right). \quad (5.56)$$

where we have defined the midpoint ghost derivative

$$\partial_{\bar{b}} A \equiv \partial_{\bar{x}^b} A = \frac{\partial A}{\partial x^b (\pi/2)}, \quad (5.57)$$

which is discussed further below. The equation of motion that follows from this action is

$$\{Q, A\}_\star + g_0 \{\partial_{\bar{b}} A, \partial_{\bar{b}} A\}_\star = 0. \quad (5.58)$$

The action (5.56) is invariant under the BRST gauge transformation given by

$$\delta_\Lambda A = [Q, \Lambda]_\star + g_0 \{\partial_{\bar{b}} A, \partial_{\bar{b}} \Lambda\}_\star, \quad (5.59)$$

where  $\Lambda$  is an arbitrary bosonic string field with ghost number  $-2$ . Then every term in this equation has total ghost number  $-1$ . The proof of this gauge symmetry,  $\delta_\Lambda S = 0$ , is given in Appendix-B.

We elaborate now on the properties of the midpoint ghost derivative  $\partial_{\bar{b}} A$ . Midpoint degrees of freedom are insensitive to our star product as explained earlier and in Appendix-A. The required midpoint insertions in our formalism, denoted by  $\partial_{\bar{b}} A$ , amounts to a derivative with respect to the midpoint ghost coordinate  $x^b(\pi/2)$ . This could also be written as a star-anticommutator with the field  $p_b(\sigma)$  at the midpoint

$$\partial_{\bar{b}} A \equiv \frac{\partial A(x, p)}{\partial x^b (\pi/2)} = -i \left\{ p_b \left( \frac{\pi}{2} \right), A(x, p) \right\}_\star. \quad (5.60)$$

Then the field  $\partial_{\bar{b}}A$  has ghost number zero and our interaction term in (5.56),  $A \star \partial_{\bar{b}}A \star \partial_{\bar{b}}A$ , has the desired ghost number  $-1$ . More generally, after doing the integral over the midpoint  $\bar{x}^b$  we obtain, with any three distinct fields,

$$\int d\bar{x}^b (A_1 \star \partial_{\bar{b}}A_2 \star \partial_{\bar{b}}A_3) = \int' \partial_{\bar{b}}A_1 \star \partial_{\bar{b}}A_2 \star \partial_{\bar{b}}A_3, \quad (5.61)$$

where  $\int'$  implies integration over the remaining modes (excluding the midpoint  $\bar{x}^b \equiv x^b(\pi/2)$ ). Hence, under the integral  $\int d\bar{x}^b$  in (5.56) there is a symmetry in moving the derivative  $\partial_{\bar{b}}$  from one field to another (noting  $\partial_{\bar{b}}^2 = 0$ ), so our interaction term in the action (5.56) is symmetric under the cyclic interchange of the fields. Hence in (5.56) there is no need for parentheses that prescribe the order of operations of the star products and/or derivatives  $\partial_{\bar{b}}$ .

It must be noted that, if the field is expressed in terms of independent ghost modes that distinguish the midpoint (see section (VIII A)),  $(\bar{x}^b, x_e^{b,c}, p_{(b,c)o})$ , where  $\bar{x}^b = x^b(\pi/2) = x^0 + \sqrt{2} \sum_e \cos \frac{e\pi}{2} x_e$  is the midpoint, then the midpoint derivative  $\partial_{\bar{b}}$  is equivalent to a derivative with respect to the single midpoint mode  $\bar{x}^b$

$$\partial_{\bar{b}}A = \frac{\partial A(x^{b,c}(\sigma), p_{b,c}(\sigma), \dots)}{\partial x^b(\pi/2)} = \partial_{\bar{x}^b} \tilde{A}(\bar{x}^b, x_e^{b,c}, p_{(b,c)o}, \dots). \quad (5.62)$$

We can use any of the expressions in (5.62) for computing  $\partial_{\bar{b}}A$  depending on the basis which is used in specific computations, i.e. the new continuous  $\sigma$  basis whose  $\star$  product does not distinguish the midpoint, or the old discrete mode basis whose  $\star$  product did distinguish the midpoint (see section (VIII A) for more details).

### E. Siegel gauge

The general field in MSFT  $A(x^M, p_M)$  may be written more explicitly in terms of the ghosts in the  $b, c$  combinations given in (5.18) as  $A(x^\mu, p_\mu, b, c)$ . We recall that in this basis  $c(\sigma, \varepsilon)$  has no zero mode while  $b(\sigma, \varepsilon)$  has a zero mode  $b_0 = \frac{1}{\pi} \int_0^\pi d\sigma b(\sigma, \varepsilon) = -ix_0^b$  (see Eq.(5.9)). The field may be expanded in powers of  $x_0^b$ , and since this is a fermion, the general expansion has only two terms

$$A = x_0^b A^{(0)} + A^{(-1)} \quad (5.63)$$

where  $A^{(0)}$  or  $A^{(-1)}$  do not contain  $x_0^b$ , and the labels  $(i)$  mean ghost number  $i = 0, -1$ .

Now we choose the Siegel gauge which satisfies  $\hat{x}_0^b A_s = 0$ , where the subscript  $s$  indicates the Siegel gauge. The zero mode ghost operator  $\hat{x}_0^b$  is given by  $\hat{x}_0^b = \left( \frac{i}{2\pi} \int_0^\pi d\sigma \left( \hat{B}(\sigma, \varepsilon) + \hat{B}(-\sigma, \varepsilon) \right) \right)$ . From the iQM representation of the QM operator  $\hat{B}$  given in Eq.(5.16), we note that the operator  $\hat{x}_0$  is diagonal on  $A(x, p)$ , so that  $\hat{x}_0^b A_s = x_0^b A_s = \left( 0 A_s^{(0)} + x_0^b A_s^{(-1)} \right) = 0$ , where we have used  $(x_0^b)^2 = 0$ . Hence, in this gauge we must have  $A_s^{(-1)} = 0$ , or

$$A_s = x_0^b A_s^{(0)}, \quad (5.64)$$

with the understanding that the zero-ghost-number field  $A_s^{(0)}$  is independent of  $x_0^b$ . In the further deliberations it is sometimes convenient to consider the midpoint coordinate  $\bar{x}^b$  instead of the zeroth mode  $x_0^b$  because the star product of the midpoint  $\bar{x}^b$  with any string field is trivial as discussed earlier and in Appendix-A. That is, taking into account  $x_0^b = \bar{x}^b + \sum_e w_e x_e^b$ , with  $w_e \equiv -\sqrt{2}(-1)^{e/2}$  as in (8.10), any field of the form (5.63) can be rewritten as:

$$A = x_0^b A^{(0)} + A^{(-1)} = \bar{x}^b A^{(0)} + \tilde{A}^{(-1)}, \text{ with } \tilde{A}^{(-1)} = \sum_e w_e x_e^b A^{(0)} + A^{(-1)}, \quad (5.65)$$

where neither  $A^{(0)}$  nor  $\tilde{A}^{(-1)}$  contain the zero mode  $x_0^b$  or the midpoint mode  $\bar{x}^b$ . Hence in the Siegel gauge

$$A_s = x_0^b A_s^{(0)} = \bar{x}^b A_s^{(0)} + \tilde{A}_s^{(-1)}, \text{ with } \tilde{A}_s^{(-1)} \equiv w_e x_e^b A_s^{(0)}. \quad (5.66)$$

We now see that the midpoint ghost derivative is  $\partial_{\bar{b}} A_s = A_s^{(0)}$ , and insert it where it occurs in the action (5.56) and the equation of motion (5.58) in the Siegel gauge

$$\partial_{\bar{b}} A_s \star \partial_{\bar{b}} A_s = A_s^{(0)} \star A_s^{(0)}. \quad (5.67)$$

To complete the Siegel gauge we also need to identify the dependence of the field  $Q(x, p)$  on the zero ghost mode  $x_0^b$ . Recall that for the QM operator  $\hat{Q}_B A$  it is well known that the  $x_0^b$  dependence is isolated as follows [29]

$$\hat{Q}_B A = \left( \hat{L}_0 - 1 \right) \partial_{x_0^b} A + \hat{Q}_1 A + x_0^b \hat{Q}_2 A, \quad (5.68)$$

where the operators  $(\hat{Q}_1, \hat{Q}_2)$ , with ghost numbers  $(1, 2)$  respectively, do not depend on  $x_0^b$ . Taking into account that we have,  $p_b(\sigma) \star A = \left( p_b(\sigma) + \frac{i}{2} \partial_{x^b(\sigma)} \right) A$  when  $\sigma \leq \pi/2$ , and

$A \star p_b(\sigma) = A \left( p_b(\sigma) - \frac{i}{2} \overleftarrow{\partial}_{x^b(\sigma)} \right)$  when  $\sigma \geq \pi/2$ , we deduce that for the field  $Q(x, p)$  we have the parallel property

$$\hat{Q}_B A = \{Q, A\}_\star = \left( \left\{ \mathcal{L}_0, \partial_{x_0^b} A \right\}_\star - \partial_{x_0^b} A \right) + \{Q_1, A\}_\star + x_0^b [Q_2, A]_\star, \quad (5.69)$$

where  $\mathcal{L}_0(x, p)$ ,  $Q_1(x, p)$ ,  $Q_2(x, p)$  are fields given below (of the corresponding ghost numbers) that do not contain any dependence on  $x_0^b$ , and applied on  $A$  with anticommutators or commutators in the iQM as determined by their ghost numbers.

Now we examine  $\{Q, A_s\}$  in the Siegel gauge  $A_s = x_0^b A_s^{(0)}$ , and after taking into account  $(x_0^b)^2 = 0$ , we find

$$\{Q, A_s\} = \left( \left\{ \mathcal{L}_0, A_s^{(0)} \right\}_\star - A_s^{(0)} \right) - x_0^b [Q_1, A_s^{(0)}]_\star, \quad (5.70)$$

where the last term is a star-commutator since  $Q_1$  is fermionic while  $A_s^{(0)}$  is bosonic.

Using the results in Eqs.(5.66-5.70) we now evaluate the action (5.56) in the Siegel gauge by using the rules of Grassmann integration,  $\int dx_0^b x_0^b = 1$  and  $\int dx_0^b = 0$ , we obtain

$$S_s = -\text{Str}' \left( A_s^{(0)} \star \left( \mathcal{L}_0 - \frac{1}{2} \right) \star A_s^{(0)} + \frac{g_0}{3} A_s^{(0)} \star A_s^{(0)} \star A_s^{(0)} \right), \quad (5.71)$$

where  $\text{Str}'$  no longer contains the integration over ghost zero mode  $x_0^b$ . Similarly we evaluate the equation of motion (5.58) in the Siegel gauge by identifying the coefficients for the zeroth power and the first power of  $x_0^b$

$$\left\{ \mathcal{L}_0, A_s^{(0)} \right\}_\star - A_s^{(0)} + g_0 A_s^{(0)} \star A_s^{(0)} = 0, \quad [Q_1, A_s^{(0)}]_\star = 0. \quad (5.72)$$

Note that the last equation is a constraint that supplements the equation of motion that follows from the gauge fixed action (5.71). The constraint amounts to applying all the remaining Virasoro constraints on the Siegel gauge field  $A_s^{(0)}$ .

To complete this section we give the explicit form of the fields  $\mathcal{L}_0(x, p)$  and  $Q_1(x, p)$  that correspond to the iQM representation of corresponding QM operators  $\hat{L}_0$  and  $\hat{Q}_1$ . The string field  $Q_1(x, p)$  is the field  $Q(x, p)$  after dropping all the effects of the zero ghost mode  $x_0^b$ . The string field  $\mathcal{L}_0(x, p)$  is obtained from Eqs.(5.36-5.39) and is given by

$$\mathcal{L}_0(x, p) = \frac{1}{\pi} \int_0^{\pi/2} d\sigma \sum_{\pm} \left( T_{\star\pm}^m + T_{\star\pm}^{gh} \right) (x_+^M, p_{-M}), \quad (5.73)$$

where the integral is over half the string, and matter  $T_{\pm}^m$  is for any CFT. To be fully explicit, we give the example of the flat CFT in  $d = 26$ , for which the expressions for  $T_{\star\pm}^m, T_{\star\pm}^{gh}$  are



similar as given in (5.26,5.33). Combining all terms,  $\mathcal{L}_0$  takes the following  $\text{SO}(25,1) \times \text{Sp}(2)$  symmetric form (normal ordering is implied)

$$\mathcal{L}_0 = \frac{1}{2\pi} \int_0^{\pi/2} d\sigma \left( \begin{array}{c} \eta_{\mu\nu} \left( \begin{array}{c} \pi^2 e^{-\varepsilon|\partial_\sigma|} p^\mu(\sigma) \star e^{-\varepsilon|\partial_\sigma|} p^\nu(\sigma) \\ + e^{-\varepsilon|\partial_\sigma|} \partial_\sigma x^\mu(\sigma, \varepsilon) \star e^{-\varepsilon|\partial_\sigma|} \partial_\sigma x^\nu(\sigma, \varepsilon) \end{array} \right) \\ + i\varepsilon_{mn} \left( \begin{array}{c} \pi^2 e^{-\varepsilon|\partial_\sigma|} \partial_\sigma p^m(\sigma) \star e^{-\varepsilon|\partial_\sigma|} \partial_\sigma p^n(\sigma) \\ + e^{-\varepsilon|\partial_\sigma|} x^m(\sigma, \varepsilon) \star e^{-\varepsilon|\partial_\sigma|} x^n(\sigma, \varepsilon) \end{array} \right) \end{array} \right). \quad (5.74)$$

The only zero modes that survive in this expression are the matter zero modes  $(x_0^\mu, p_{0\mu})$ . Since the star product is symmetric under  $\text{OSp}(d|2)$ , the cubic term in the action (5.71) is supersymmetric under transformations that mix matter and ghost degrees of freedom even when we have any CFT with curved backgrounds. However, the supersymmetry is broken by the quadratic term because  $\mathcal{L}_0$  is not symmetric under  $\text{OSp}(26|2)$ : For a non-trivial CFT the curved background in  $\mathcal{L}_0$  breaks even the linear  $\text{SO}(d)$ ; for the flat CFT  $\text{SO}(26)$  is valid in  $\mathcal{L}_0$  but  $\text{OSp}(26|2)$  is broken since in Eq.(5.74) the  $\partial_\sigma$  derivatives are applied on the ghost momenta  $p^m$  rather than on ghost positions  $x^m$ . However, we can bring the expression (5.74) to the expected supersymmetric form with a simple change of the basis in the ghost sector. This will be discussed in the next section (V F).

We have reached the stage where we can now make direct contact in detail with the MSFT formulation using old star  $\star$  product for the flat CFT and distinguishing the midpoint. This is important because we can then claim that all the previous successful computations are now also a direct consequence of the more general new formalism. The correspondence between the old and new formulations is obtained through the Siegel gauge action (5.71) and the discussion in Appendix-A about the relation between the various bases of the degrees of freedom, the new  $\sigma$ -basis, the new mode basis including the center of mass mode  $x_0$ , and the old mode basis including the midpoint mode,

$$A(x(\sigma), p(\sigma)) = A(x_0, x_e, p_o) = \tilde{A}(\bar{x}, x_e, p_o), \quad (5.75)$$

assuming that the ghost zero mode  $x_0^b$  or the corresponding midpoint mode  $\bar{x}^b$  is already integrated out. The only remaining zero mode is the matter zero mode  $x_0^\mu$ . Then the appropriate star product in  $A(x_0, x_e, p_o)$  basis takes the form given in Eqs.(8.25,8.26). As discussed in Appendix-A this reproduces all the results obtained in the old basis  $\tilde{A}(\bar{x}, x_e, p_o)$  [2]-[5]. Hence we have reproduced all of the previous work based on the flat CFT in the Siegel

gauge. We are now prepared to tackle non-perturbative computations with a more efficient tool.

### F. $\text{OSp}(d|2)$ Supersymmetry Acting on Matter and Ghosts

In the discussion above we noted that the star product has an  $\text{OSp}(d|2)$  matter-ghost supersymmetry for any CFT, but the stress tensors for matter versus ghosts as well as the ghost structure of the BRST operator break this supersymmetry. So, the kinetic term of the SFT action (5.56) breaks the supersymmetry while the interactions are supersymmetric.

Consequently, in Feynman-like diagram computations in the Siegel gauge, in the flat CFT case, the breaking of  $\text{OSp}(d|2)$  is in the propagator and not in the interactions. Since the breaking of the supersymmetry amounts to moving the  $\partial_\sigma$  derivative from the ghost  $P^m$  to the ghost  $X^m$  in the expression for  $\hat{L}_0$  in (5.34) or  $\mathcal{L}_0$  in (5.74), we can construct a simple algorithm to simplify all computations as if there is  $\text{OSp}(d|2)$  symmetry in the full theory, and then modify the final computation with a simple rule that takes into account this breaking of the symmetry in propagators. This algorithm was noted and used efficiently in past computations in MSFT [3][4][5] and will be illustrated below with an example.

However, we discovered that there is a better approach: it is possible to rewrite the theory in a slightly modified basis of the ghost degrees of freedom such as to display fully the  $\text{OSp}(d|2)$  symmetry in the Siegel gauge (but not in the general gauge) including in the kinetic term or the propagators. We may then use supersymmetric propagators in all computations in the Siegel gauge. The final results in computations, such as amplitudes, are the same as before [3][4][5].

To show how this works we need to review the ghost sector and show that there is a more general way to extract the ghost phase space operators  $(\hat{X}^m, \hat{P}_m)$  from the ghost  $(\hat{B}, \hat{C})$  operators. Namely, instead of Eq.(5.7), we can introduce a more general formula which includes a parameter  $\alpha$  as follows

$$\begin{aligned}\hat{B}(\pm\sigma) &= \left(-i\hat{X}^b(\sigma) \pm \pi|\partial_\sigma|^{\alpha-1}\partial_\sigma\hat{P}_c(\sigma)\right), \\ \hat{C}(\pm\sigma) &= \left(\pi\hat{P}_b(\sigma) \mp i|\partial_\sigma|^{-\alpha-1}\partial_\sigma\hat{X}^c(\sigma)\right).\end{aligned}\tag{5.76}$$

Taking  $\alpha = 1$  reproduces (5.7). Results of any computations should be independent of  $\alpha$  since this is only a rewriting of the same  $(\hat{B}, \hat{C})$  operators. Indeed we have checked that

physical quantities, such as amplitudes, are independent of  $\alpha$ . So we may explore if the parameter  $\alpha$  leads to some interesting consequences resulting from this rearrangement of the degrees of freedom. The answer is yes: as it will be pointed out below, choosing  $\alpha = -1$  rather than  $\alpha = 1$ , will be useful to display a supersymmetry between matter and ghost degrees of freedom in the Siegel gauge.

For the more general definition (5.76), the modes in Eqs.(5.9,5.10) generalize to the following form including the parameter  $\alpha$

$$\hat{X}_0^b = i\hat{b}_0, \quad \hat{X}_{n \geq 1}^b = \frac{i}{\sqrt{2}} \left( \hat{b}_n + \hat{b}_{-n} \right), \quad \hat{X}_{n \geq 1}^c = -\frac{n^\alpha}{\sqrt{2}} (\hat{c}_n - \hat{c}_{-n}), \quad (5.77)$$

$$\hat{P}_{b0} = \hat{c}_0, \quad \hat{P}_{b,n \geq 1} = \frac{1}{\sqrt{2}} (\hat{c}_n + \hat{c}_{-n}), \quad \hat{P}_{c,n \geq 1} = \frac{i}{\sqrt{2}n^\alpha} (\hat{b}_n - \hat{b}_{-n}), \quad (5.78)$$

and similarly for the inverse relations

$$\hat{b}_0 = -i\hat{X}_0^b, \quad \hat{b}_{n \geq 1} = \frac{i}{\sqrt{2}} \left( -\hat{X}_{n \geq 1}^b - n^\alpha \hat{P}_{c,n \geq 1} \right), \quad \hat{b}_{(-n \leq 1)} = \frac{i}{\sqrt{2}} \left( -\hat{X}_{n \geq 1}^b + n^\alpha \hat{P}_{c,n \geq 1} \right), \quad (5.79)$$

$$\hat{c}_0 = \hat{P}_{b0}, \quad \hat{c}_{n \geq 1} = \frac{1}{\sqrt{2}} \left( -n^{-\alpha} \hat{X}_{n \geq 1}^c + \hat{P}_{b,n \geq 1} \right), \quad \hat{c}_{(-n \leq 1)} = \frac{1}{\sqrt{2}} \left( n^{-\alpha} \hat{X}_{n \geq 1}^c + \hat{P}_{b,n \geq 1} \right). \quad (5.80)$$

Then, the regulated expression in (5.14) are generalized as

$$\hat{B}(\pm\sigma, \varepsilon) = \left( -ie^{-\varepsilon|\partial_\sigma|} \hat{X}^b(\sigma, \varepsilon) \pm \pi |\partial_\sigma|^{\alpha-1} \partial_\sigma \hat{P}_c(\sigma) \right), \quad (5.81)$$

$$\hat{C}(\pm\sigma, \varepsilon) = \left( \pi \hat{P}_b(\sigma) \mp ie^{-\varepsilon|\partial_\sigma|} |\partial_\sigma|^{-\alpha-1} \partial_\sigma \hat{X}^c(\sigma, \varepsilon) \right), \quad (5.82)$$

The ghost stress tensor Eq.(5.33) is also generalized to include the effects of  $\alpha$

$$\hat{T}_{\pm\pm}^{gh} = \frac{1}{4} (i\epsilon_{mm'}) : \left( \begin{array}{c} \pi |\partial_\sigma|^{\frac{\alpha-1}{2}} \partial_\sigma \hat{P}^m(\sigma) \\ \mp e^{-\varepsilon|\partial_\sigma|} |\partial_\sigma|^{\frac{1-\alpha}{2}} \hat{X}^m(\sigma, \varepsilon) \end{array} \right) \left( \begin{array}{c} \pi |\partial_\sigma|^{\frac{\alpha-1}{2}} \partial_\sigma \hat{P}^{m'}(\sigma) \\ \mp e^{-\varepsilon|\partial_\sigma|} |\partial_\sigma|^{\frac{1-\alpha}{2}} \hat{X}^{m'}(\sigma, \varepsilon) \end{array} \right) : -\frac{1}{2} i \partial_{\pm\sigma} J_{\pm}^{gh} \quad (5.83)$$

while the zero mode Virasoro operator in (5.34) generalizes to the following form

$$\hat{L}_0 = \frac{1}{\pi} \int_0^\pi d\sigma \left( \begin{array}{c} \frac{1}{2} \eta_{\mu\nu} \left( \pi^2 \hat{P}^\mu \hat{P}^\nu + \hat{X}^\mu |\partial_\sigma|^2 \hat{X}^\nu \right) \\ + \frac{1}{2} i \epsilon_{mn} \left( \pi^2 \hat{P}^m |\partial_\sigma|^{1+\alpha} \hat{P}^n + \hat{X}^m |\partial_\sigma|^{1-\alpha} \hat{X}^n \right) \end{array} \right). \quad (5.84)$$

Now it is evident that for  $\alpha = -1$  (as opposed to  $\alpha = 1$  in (5.34)) this expression has the same form for both the matter and ghost parts. In fact, for  $\alpha = -1$  this displays an  $\text{OSp}(d|2)$  supersymmetry between matter and ghost degrees of freedom in the operator  $\hat{L}_0$  that determines the kinetic term in the Siegel gauge. Recall that the interaction terms are

already supersymmetric. Therefore computations in the Siegel gauge can now be performed in a supersymmetric fashion at every step of any computation provided we adopt the new expressions for  $\hat{L}_0$  given above with  $\alpha = -1$ .

Similarly we record here the iQM version of the BRST operator for the case  $\alpha = -1$  for the flat CFT (which is different than the corresponding expressions in (5.40) or (5.74))

$$Q(x, p) = \frac{1}{2\pi} \int_0^{\pi/2} d\sigma \left[ \begin{array}{l} \pi p_b (\pi^2 p^\mu p_\mu + x'^\mu x'_\mu + i\pi^2 p_c p_b + i x'^b x'^c) \\ -i\pi (|\partial_\sigma|^{-2} x'^c) (2x'^\mu p_\mu + x'^c p_c + x'^b p_b) \end{array} \right]. \quad (5.85)$$

where  $x'^M \equiv \partial_\sigma x^M$ . This  $Q(x, p)$  is not supersymmetric, and hence the kinetic term in the general gauge is not supersymmetric; However, the kinetic term in the Siegel gauge becomes accidentally  $\text{OSp}(d|2)$  invariant after integrating out the  $x_0^b$  mode as described above.

### G. Effective non-Perturbative Purely Cubic Quantum Action

Our discussion of the Siegel gauge so far is at the classical field theory level, so in the action (5.71) we have the zero ghost number field  $A_s^{(0)}$ . However, in a quantum version of string field theory the path integral includes Faddeev-Popov ghosts. In this case there are also ghosts of ghosts of  $b, c$  types ad infinitum [30]. Including all of these additional ghost fields, the quantum action can be written in a convenient notation. The full quantum SFT effective action takes a similar form to (5.56), but now the string field includes all positive and negative ghost numbers,  $A = \sum A^{(i)}$ , not only the classical fields  $A_s^{(0)}$ , so the effective action in the path integral in the Siegel gauge contains this generalized  $A$ , and in our description takes the form

$$S_{eff} = -\text{Str} \left( A \star Q \star A + \frac{g_0}{3} A \star A \star A \right). \quad (5.86)$$

It is interesting to note that, after using  $Q \star Q = 0$ , this action may be rewritten in the purely cubic form

$$S_{eff}(\bar{A}) = -\frac{1}{3g_0^2} \text{Str}(\bar{A} \star \bar{A} \star \bar{A}), \text{ with } \bar{A} \equiv g_0 A + Q. \quad (5.87)$$

This action is invariant under the general gauge transformation

$$\delta_\Lambda \bar{A} = [\bar{A}, \Lambda]_\star = \bar{A} \star \Lambda - (-1)^{A\Lambda} \Lambda \star \bar{A}. \quad (5.88)$$

where  $\Lambda$  includes all ghost numbers, just like  $A$  does. From this we see that (5.86) is also invariant by substituting  $\bar{A}$  in terms of  $A$  as in (5.87).

The purely cubic version has been noted before by many authors [31] as a formal property of SFT for the open string. But in many treatments various anomalies, including associativity anomalies [3][7], emerged that could not be satisfactorily resolved, so as to cast doubt on the utility of this observation. Indeed not having a satisfactory resolution of anomalies leads to wrong conclusions [3]. In our formalism we have introduced a reliable regulator  $\varepsilon$  as part of the definition of the new MSFT. With this regulator there are no anomalies and this allows us to use the purely cubic form of the action reliably.

Thus we will take the fundamental *non-perturbative form of regulated MSFT* to be of the purely cubic form (5.87), including our reliable regulator  $\varepsilon$  discussed throughout this paper. This form of the action has some remarkable properties as follows

- The  $\star$  is background independent, it does not depend on any specific CFT, it is defined only by an abstract phase space. Similarly, the field  $\bar{A}(x, p)$  is independent of any CFT.
- This action has a huge amount of symmetry because all supercanonical transformations of  $(x, p)$  leave the Moyal product invariant. The action is invariant when the field  $\bar{A}$  transforms under a similarity transformation in the iQM as follows

$$\bar{A} \rightarrow \bar{A}' = U \star \bar{A} \star U^{-1}, \text{ where } U = (e^{\varepsilon(x, p)})_{\star} \text{ with any } \varepsilon(x, p), \quad (5.89)$$

where  $\varepsilon(x, p)$  is regarded as a generator of supercanonical transformations on matter and ghosts in the iQM.

- A subset of supercanonical transformations is a finite global subset of super rotations  $\text{OSp}(d|2)$  that act linearly on the supervectors  $(x^M, p_M)$ , namely  $x^M \rightarrow (Sx)^M$  and  $p_M \rightarrow (S^{-1}p)_M$ , with  $S \in \text{OSp}(d|2)$  (more accurately  $\text{OSp}((d-1, 1)|2)$ ). These supertransformations mix the matter and ghost degrees of freedom. The Moyal  $\star$  in (2.9) and the phase space integration measure (5.46) are manifestly invariant under this  $\text{OSp}(d|2)$ . Hence when  $\bar{A}(x, p)$  is transformed as  $\bar{A}(x, p) \rightarrow \bar{A}(Sx, S^{-1}p)$  the action is invariant. When the action is rewritten in terms of  $Q$  and  $A$  as in Eq.(5.86), the matter-ghost symmetry is spontaneously broken (i.e. hidden); it is still manifest in the cubic term but broken in the quadratic term because  $Q$  is not manifestly symmetric.

Keeping track of the broken symmetry is easy and it turns out to be valuable because this simplifies computations (see below).

- Rewriting the purely cubic theory back into the form in Eq.(5.86) is analogous to spontaneous breakdown. It is a rearrangement of the non-perturbative theory into a perturbative expansion around a classical solution of the non-perturbative equation of motion

$$\bar{A}(x, p) \star \bar{A}(x, p) = 0. \quad (5.90)$$

The BRST field  $Q(x, p)$  associated to any CFT, as given in Eqs.(5.39) is an exact solution of this equation  $\bar{A}_{sol}(x, p) = Q(x, p)$ . *Our approach for the construction of  $Q(x, p)$  for any CFT given in the previous section provides an infinite number of solutions to Eq.(5.90), namely one for each exact conformal CFT.* The perturbative expansion of the field in power of  $g_0$  around this solution is  $\bar{A}(x, p) = Q(x, p) + g_0 A(x, p)$ . Inserting this in the non-perturbative action (5.87) produces the perturbative setup in Eq.(5.86) or Eq.(5.56).

- Although the non-perturbative theory is background independent, the perturbative expansion  $\bar{A}(x, p) = Q(x, p) + g_0 A(x, p)$  is obviously dependent on the background field  $Q(x, p)$  that defines the CFT associated to the choice of the solution  $Q(x, p)$ . All CFTs correspond to solutions of the non-perturbative equation. But it is not clear if all solutions of (5.90) are CFTs.
- Using the observation in the previous paragraph we can now obtain a large class of non-perturbative solutions to the standard string field theory equation of motion  $\hat{Q}A + g_0 A \star A = 0$ , namely, since this equation may be written as  $(Q_1(x, p) + g_0 A(x, p))^2 = 0$  where  $Q_1(x, p)$  is associated with some CFT<sub>1</sub>, we can give a solution for  $A$  in the form

$$g_0 A(x, p) = Q_2(x, p) - Q_1(x, p) \quad (5.91)$$

where  $Q_2(x, p)$  is another CFT<sub>2</sub> that is constructed from the same degrees of freedom  $(x, p)$ . In principle there are an infinite number of solutions. In practice, going over pairs of exactly conformal CFTs that we know how to handle (such as those similar to [23]-[26]), Eq.(5.91) provides a non-perturbative explicit solution to string field theory.

## VI. ILLUSTRATIONS WITH FLAT SPACE CFT

In this section we will illustrate some computations when the CFT corresponds to the flat Minkowski background in  $d = 26$ . Our goal here is to make our notation transparent to the reader by showing how to proceed in explicit simple computations using our formalism.

We have seen in Eqs.(5.26,5.33) with  $\alpha = -1$  that in this case the total stress tensor is  $\text{OSp}(26|2)$  invariant and has the form

$$\hat{T}_{\pm\pm}^{m+gh}(\sigma, \varepsilon) = \frac{1}{4} \left[ \pi \hat{P}(\sigma) \pm e^{-\varepsilon|\partial_\sigma|} \partial_\sigma \hat{X}(\sigma, \varepsilon) \right]^2, \quad (6.1)$$

where it is implied that the indices on  $\hat{P}^M, \hat{X}^M$  are summed by using the metric for  $\text{OSp}(26|2)$ ,

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & i\epsilon_{mn} \end{pmatrix}, \quad (6.2)$$

where  $\epsilon_{mn}$  is given in Eq.(5.32). In the limit  $\varepsilon \rightarrow 0$  this  $\hat{T}_{\pm\pm}^{m+gh}(\sigma, \varepsilon)$  reduces to the usual flat-space Virasoro operators if rewritten in terms of free string modes including ghosts.

### A. Oscillators in $\sigma$ -space and Perturbative Vacuum

It is useful to define regulated creation/annihilation operators in  $\sigma$  space as follows. In  $\varepsilon \rightarrow 0$  limit these are equivalent to the standard string oscillators in mode space and are given by

$$\hat{a}_\pm^M(\sigma, \varepsilon) = \frac{1}{\sqrt{2}} \left( \pi \hat{P}_N(\sigma) \eta^{NM} \pm i e^{-\varepsilon|\partial_\sigma|} |\partial_\sigma| \hat{X}^M(\sigma, \varepsilon) \right). \quad (6.3)$$

The inverse relation is

$$\pi \hat{P}_M(\sigma) = \frac{1}{\sqrt{2}} \left( \hat{a}_+^N(\sigma, \varepsilon) + \hat{a}_-^N(\sigma, \varepsilon) \right) g_{NM}, \quad (6.4)$$

$$e^{-\varepsilon|\partial_\sigma|} \hat{X}^M(\sigma, \varepsilon) = \frac{-i}{\sqrt{2}|\partial_\sigma|} \left( \hat{a}_+^M(\sigma, \varepsilon) - \hat{a}_-^M(\sigma, \varepsilon) \right). \quad (6.5)$$

Note that in comparing Eqs.(6.3,6.1) one should distinguish  $|\partial_\sigma| \equiv \sqrt{-\partial_\sigma^2}$  from  $\partial_\sigma$ . These  $\hat{a}_\pm^M(\sigma, \varepsilon)$  satisfy (using  $[\hat{X}^M, \hat{P}_N] = i\delta_N^M$ )

$$[\hat{a}_-^M(\sigma, \varepsilon), \hat{a}_+^N(\sigma', \varepsilon)] = \pi e^{-\varepsilon|\partial_\sigma|} |\partial_\sigma| \delta_\varepsilon(\sigma, \sigma') g^{MN}. \quad (6.6)$$

In terms of  $\hat{a}_\pm^M(\sigma, \varepsilon)$  the Virasoro operator  $\hat{L}_0^{m+gh}$  takes the normal ordered form

$$\hat{L}_0^{m+gh} = \frac{1}{2\pi} \int_0^\pi \sum_\pm : \hat{T}_{\pm\pm}^{m+gh}(\sigma, \varepsilon) : = \frac{g_{MN}}{\pi} \int_0^\pi \hat{a}_+^M(\sigma, \varepsilon) \hat{a}_-^N(\sigma, \varepsilon). \quad (6.7)$$

So, the vacuum state in position space, which satisfies  $\hat{a}_-^N(\sigma, \varepsilon) \Psi_0(X(\sigma, \varepsilon)) = 0$ , is given by the Gaussian

$$\Psi_0(X(\cdot, \varepsilon)) \sim \exp \left\{ -\frac{g_{MN}}{2\pi} \int_0^\pi d\sigma X^M(\sigma, \varepsilon) |\partial_\sigma| X^N(\sigma, \varepsilon) \right\}. \quad (6.8)$$

In the limit  $\varepsilon \rightarrow 0$  this reduces to the expected familiar vacuum state in the oscillator basis expressed in position space as a Gaussian. By using the derivative representation of  $\hat{P}_M(\sigma)$  in position space (see Eq.3.14),  $\hat{P}_M(\sigma) \Psi(X) = -ie^{-\varepsilon|\partial_\sigma|} (\partial\Psi/\partial X_M(\sigma, \varepsilon))$ , one can verify that indeed  $\hat{a}_-^N(\sigma, \varepsilon) \Psi_0(X(\sigma, \varepsilon)) = 0$  is satisfied for both matter and ghosts. Hence  $\hat{L}_0^{m+gh} \Psi_0(X(\cdot, \varepsilon)) = 0$  so that  $\Psi_0(X(\cdot, \varepsilon))$  is indeed the perturbative vacuum state in position space.

Now we turn to the field in the Moyal space  $A(x, p)$ . The star representation of the creation-annihilation operators above are obtained by using the prescription in Eq.(5.2).

$$\hat{a}_\pm^M(\sigma, \varepsilon) A(x, p) = \begin{cases} a_\pm^M(\sigma, \varepsilon) \star A(x, p), & \text{if } 0 \leq \sigma \leq \pi/2, \\ A(x, p) \star a_\pm^M(\sigma, \varepsilon) (-1)^{MA}, & \text{if } \pi/2 \leq \sigma \leq \pi, \end{cases} \quad (6.9)$$

where  $a_\pm^M(\sigma, \varepsilon)$  (without the hat  $\hat{\phantom{a}}$ ) are string fields, constructed from  $(x, p)$ , just like  $A(x, p)$

$$a_\pm^M(\sigma, \varepsilon) = \frac{1}{\sqrt{2}} e^{-\varepsilon|\partial_\sigma|} (\pi p_N(\sigma) g^{NM} \pm i |\partial_\sigma| x^M(\sigma, \varepsilon)). \quad (6.10)$$

The vacuum state is identified as the field  $A_0(x, p)$  that is annihilated by  $a_-^M(\sigma, \varepsilon)$  either from the left or the right under star products,

$$a_-^M(\sigma, \varepsilon) \star A_0(x, p) = 0, \quad \text{if } 0 \leq \sigma \leq \pi/2, \quad (6.11)$$

$$A_0(x, p) \star a_-^M(\sigma, \varepsilon) = 0, \quad \text{if } \pi/2 \leq \sigma \leq \pi. \quad (6.12)$$

The solution is

$$A_0(x, p) = \mathcal{N}_0 \exp \left\{ -\frac{1}{2} \int_0^\pi d\sigma \left( g_{MN} x^M(\sigma, \varepsilon) \frac{|\partial_\sigma|}{\pi} x^N(\sigma, \varepsilon) + g^{MN} p_M(\sigma) \frac{\pi}{|\partial_\sigma|} p_N(\sigma) \right) \right\}. \quad (6.13)$$

As expected, this  $A_0(x_+, p_-)$  is consistent with the Fourier transform of the position space field  $\Psi_0(X) = \Psi_0(x_+, x_-)$  with respect to the variable  $x_-(\sigma, \varepsilon)$ . Note that the center of mass momentum for matter vanishes on the vacuum field

$$\begin{aligned} \hat{P}_\mu^{cm} A_0 &= \int_0^{\pi/2} d\sigma e^{-\varepsilon|\partial_\sigma|} p_\mu(\sigma) \star A_0 + \int_{\pi/2}^\pi d\sigma A_0 \star e^{-\varepsilon|\partial_\sigma|} p_\mu(\sigma) \\ &= \int_0^\pi d\sigma e^{-\varepsilon|\partial_\sigma|} \left( \frac{1}{2} \frac{-i\partial A_0}{\partial x^\mu(\sigma, \varepsilon)} \right) + 0 \\ &= \frac{i}{2} A_0 \int_0^\pi d\sigma e^{-\varepsilon|\partial_\sigma|} \frac{|\partial_\sigma|}{\pi} x_\mu(\sigma, \varepsilon) = 0. \end{aligned} \quad (6.14)$$



In the second line the “0” represents the fact that the non-derivative piece in the star product drops out because  $p_\mu(\sigma)$  is odd under reflections from  $\pi/2$ . In the last line the integral vanishes since  $|\partial_\sigma| x_\mu(\sigma, \varepsilon)$  has no zero mode while the integrals over the remaining even modes vanish  $\int_0^\pi d\sigma \cos e\sigma = 0$ .

The normalization  $N_0$  in Eq.(6.13) is determined by demanding

$$1 = \text{Str}(A_0 \star A_0) = (N_0)^2 \int Dx Dp \exp \left\{ - \int_0^\pi d\sigma \left( g_{MN} x^M(\sigma, \varepsilon) \frac{|\partial_\sigma|}{\pi} x^N(\sigma, \varepsilon) + g^{MN} p_M(\sigma) \frac{\pi}{|\partial_\sigma|} p_N(\sigma) \right) \right\}. \quad (6.15)$$

The Gaussian integral gives determinants and this fixes  $N_0$  as follows

$$N_0 = \left( \det \left( |\partial_\sigma|^{-1/2} \right)_+ \det \left( |\partial_\sigma|^{1/2} \right)_- \right)^{-(d-2)/2}, \text{ with } d = 26. \quad (6.16)$$

The reason for  $(d-2)$  is because the integral in the bosonic sector contributes the  $(d)$  and the Grassmannian integral in the fermionic ghost sector contributes the  $(-2)$ . Another way of thinking about this is that we have a superdeterminant in the space of  $\text{OSp}(d|2)$  and this is why we get  $(d-2)$  rather than  $d+2$ . The determinants are to be evaluated in the even and odd sectors since  $x_+^M(\sigma, \varepsilon)$  has only even modes and  $p_{-M}(\sigma)$  has only odd modes. The result is<sup>3</sup>

$$\det(|\partial_\sigma|)_+ \det(|\partial_\sigma|^{-1})_- = \frac{2 \cdot 4 \cdot 8 \cdots 2n \cdots}{1 \cdot 3 \cdot 5 \cdots (2n+1) \cdots} = \sqrt{\pi/2}. \quad (6.18)$$

Therefore

$$N_0 = (\pi/2)^{(d-2)/8} = (\pi/2)^3, \text{ with } d = 26. \quad (6.19)$$

Often we are interested in the vacuum expectation values of the basic operators  $\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma)$ . These are computed easily by using the oscillator expressions in Eqs.(6.4,6.5)

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<sup>3</sup> The computation is ambiguous because the product of the eigenvalues of  $|\partial_\sigma|$  can be arranged as  $\lim_{N \rightarrow \infty} \prod_{n=1}^N \left( \frac{2n}{(2n-1)} \right) = \infty$ , or  $\lim_{N \rightarrow \infty} \prod_{n=1}^N \left( \frac{(2n)}{(2n+1)} \right) = 0$ . To get a well defined result we take the product of these two and take the square root

$$\lim_{N \rightarrow \infty} \left( \prod_{n=1}^N \left( \frac{2n}{(2n-1)} \right) \left( \frac{2n}{(2n+1)} \right) \right)^{1/2} = \sqrt{\pi/2}. \quad (6.17)$$

and the properties of the vacuum state. For example, for  $0 < \sigma_1, \sigma_2 < \pi/2$ ,

$$\begin{aligned}
& \langle e^{-\varepsilon|\partial_{\sigma_1}|} \hat{X}^M(\sigma_1, \varepsilon) e^{-\varepsilon|\partial_{\sigma_2}|} \hat{X}^N(\sigma_2, \varepsilon) \rangle \\
&= -\frac{-i}{\sqrt{2}|\partial_{\sigma_1}|} \frac{-i}{\sqrt{2}|\partial_{\sigma_2}|} \frac{\text{Str}(A_0 \star (\hat{a}_-^M(\sigma_1, \varepsilon) \hat{a}_+^N(\sigma_2, \varepsilon) A_0))}{\text{Str}(A_0 \star A_0)} \\
&= \frac{\pi}{2} e^{-\varepsilon|\partial_{\sigma_1}|} e^{-\varepsilon|\partial_{\sigma_2}|} |\partial_{\sigma_2}|^{-1} \delta(\sigma_1, \sigma_2) g^{MN}
\end{aligned} \tag{6.20}$$

and similarly for vacuum expectation values that involve  $\hat{P}_M(\sigma)$  we find

$$\langle e^{-\varepsilon|\partial_{\sigma_1}|} \hat{X}^M(\sigma_1, \varepsilon) \hat{P}_N(\sigma_2) \rangle = i \frac{\pi}{2} e^{-\varepsilon|\partial_{\sigma_1}|} e^{-\varepsilon|\partial_{\sigma_2}|} \delta(\sigma_1, \sigma_2) \delta_N^M \tag{6.21}$$

$$\langle \hat{P}_M(\sigma_1) \hat{P}_N(\sigma_2) \rangle = \frac{\pi}{2} e^{-\varepsilon|\partial_{\sigma_1}|} e^{-\varepsilon|\partial_{\sigma_2}|} \delta(\sigma_1, \sigma_2) g_{MN} \tag{6.22}$$

Using these expressions we can use Wick's theorem (or equivalently operator products) to rewrite products of operators  $\hat{X}$ 's or  $\hat{P}$ 's in terms of normal ordered products. The exact parallel steps is available in the Moyal formulation, but these products occur as star products either on the left side or the right side of the field  $A(x, p)$ , for example if both  $\sigma_1, \sigma_2$  are less than  $\pi/2$

$$\hat{O}(\sigma_1) \hat{O}(\sigma_2) A(x, p) = O_\star(\sigma_1) \star O_\star(\sigma_2) \star A(x, p). \tag{6.23}$$

Wick's theorem or operator products computed in familiar operator language have the exact parallel in the Moyal formalism and therefore the c-number coefficients are identical in either formalism, as in Eq.(5.5). Therefore we can borrow well known results for quantum operator products in the CFT and use them directly for the star products of the corresponding fields in the induced iQM.

The expression for the matter vacuum (6.13) includes the ghost contribution

$$A_0^{ghost}(x, p) = N_0^{gh} \exp \left\{ -i \int_0^\pi d\sigma \left( x^b(\sigma, \varepsilon) \frac{|\partial_\sigma|}{\pi} x^c(\sigma, \varepsilon) + p_c(\sigma) \frac{\pi}{|\partial_\sigma|} p_b(\sigma) \right) \right\}. \tag{6.24}$$

which we discuss a bit more to emphasize that this is the ghost  $SL(2, \mathbb{R})$  vacuum. It is known that there are two ghost vacuum states  $A_{0\pm}$  that by definition satisfy the following relations:

$$\hat{b}_0 A_{0+} = A_{0-}, \quad \hat{c}_0 A_{0-} = A_{0+}, \tag{6.25}$$

while

$$\hat{b}_0 A_{0-} = \hat{c}_0 A_{0+} = 0. \tag{6.26}$$

Their ghost numbers should differ by one:

$$N_{gh}(A_{0+}) = N_{gh}(A_{0-}) + 1, \quad (6.27)$$

and their star product satisfies (recall Str has ghost number 1):

$$\text{Str}(A_{0+} \star A_{0-}) = \text{Str}\left(A_{0+} \star \left(\hat{b}_0 A_{0+}\right)\right) = \text{Str}(A_{0-} \star (\hat{c}_0 A_{0-})) = 1, \quad (6.28)$$

$$\text{Str}(A_{0+} \star A_{0+}) = \text{Str}(A_{0-} \star A_{0-}) = 0. \quad (6.29)$$

Therefore, we can conclude that the ghost number of  $A_{0-}$  is  $-1$ , while  $N_{gh}(A_{0+}) = 0$ . The  $SL(2, \mathbb{R})$  vacuum state has the ghost number 0, therefore it can be considered as the vacuum state  $A_{0+}$ .

Hence, the ghost vacua are the following

$$A_{0+}(x, p) = N_0^{gh} \exp \left\{ -i \int_0^\pi d\sigma \left( x^b(\sigma, \varepsilon) \frac{|\partial_\sigma|}{\pi} x^c(\sigma, \varepsilon) + p_c(\sigma) \frac{\pi}{|\partial_\sigma|} p_b(\sigma) \right) \right\}, \quad (6.30)$$

$$A_{0-}(x, p) = -i N_0^{gh} x_0^b \exp \left\{ -i \int_0^\pi d\sigma \left( x^b(\sigma, \varepsilon) \frac{|\partial_\sigma|}{\pi} x^c(\sigma, \varepsilon) + p_c(\sigma) \frac{\pi}{|\partial_\sigma|} p_b(\sigma) \right) \right\}. \quad (6.31)$$

This works correctly with the normalization conditions above (recall Str includes the Grassmannian integral  $\int dx_0^b$ ). In the Siegel gauge (5.64) we had,  $A_s = x_0^b A_s^{(0)}$ , with  $A_s^{(0)}$  not containing the ghost zero mode  $x_0^b$ . Hence  $A_s^{(0)} \sim A_{0+}^{gh}$  matches all the properties as the  $SL(2, R)$  invariant vacuum.

## B. The Monoid in the $\sigma$ -basis

A very useful tool for computations in MSFT is the monoid algebra developed in [3][4]. This arises as follows. We have seen above that the perturbative vacuum state  $A_0$  is the Gaussian string field (6.13). Excited perturbative string states are represented by the same Gaussian field multiplied by polynomials of  $(x, p)$ . The polynomials in perturbative states can be generated by taking derivatives with respect to the shift  $\lambda_x$  or  $\lambda_p$  of a shifted gaussian  $A \sim \exp(-xx - pp + \lambda_x x + \lambda_p p)$ , and then setting the  $\lambda$ 's to zero. On the other hand, at least some non-perturbative states are also shifted gaussian-like states, but with a different quadratic and linear exponent than the one for the perturbative states. This suggests that fields of the shifted-gaussian form are very common in explicit computations. It was found in [3][4] that they have nice mathematical properties that are directly useful in the computation of amplitudes, including the Veneziano amplitude [5].

The set of shifted gaussians of interest are of the form

$$A_{\mathcal{N},M,\lambda} = \mathcal{N} e^{-\xi^i M_{ij} \xi^j - \xi^i \lambda_i}. \quad (6.32)$$

where the  $\xi^i$  stand for  $(x, p)$  and the symbols  $M_{ij}, \lambda_i$  are parameters, while  $\mathcal{N}$  is a normalization. They close under the Moyal star product as follows

$$A_{\mathcal{N}_1, M_1, \lambda_1} \star A_{\mathcal{N}_2, M_2, \lambda_2} = A_{\mathcal{N}_{12}, M_{12}, \lambda_{12}}, \quad (6.33)$$

The  $\xi^i$  are a set of non-commutative (super)variables (matter and ghost phase spaces) which satisfy

$$[\xi^i, \xi^j]_\star = s^{ij}, \quad (6.34)$$

with  $s^{ij}$  a constant matrix which is antisymmetric when both  $i$  and  $j$  are bosonic and is symmetric when both  $i$  and  $j$  are fermionic. The  $(\mathcal{N}_{12}, M_{12}, \lambda_{12})$  are computed from  $(\mathcal{N}_1, M_1, \lambda_1)$  and  $(\mathcal{N}_2, M_2, \lambda_2)$  as follows. Given the data for  $(\mathcal{N}_1, M_1, \lambda_1)$  and  $(\mathcal{N}_2, M_2, \lambda_2)$  we first define the matrices  $m$

$$m_1 = M_1 s, \quad m_2 = M_2 s, \quad m_{12} = M_{12} s; \quad (6.35)$$

then the result for  $m_{12}, \lambda_{12}, \mathcal{N}_{12}$  takes the form [2][3][4]

$$m_{12} = (m_1 + m_2 m_1) (1 + m_2 m_1)^{-1} + (m_2 - m_1 m_2) (1 + m_1 m_2)^{-1}, \quad (6.36)$$

$$\lambda_{12} = (1 - m_1) (1 + m_2 m_1)^{-1} \lambda_2 + (1 + m_2) (1 + m_1 m_2)^{-1} \lambda_1 \quad (6.37)$$

$$\mathcal{N}_{12} = \frac{\mathcal{N}_1 \mathcal{N}_2}{\text{sdet} (1 + m_2 m_1)^{1/2}} e^{\frac{1}{4} ((\lambda_1 + \lambda_2)(M_1 + M_2)^{-1} (\lambda_1 + \lambda_2) - \bar{\lambda}_{12} (M_{12})^{-1} \lambda_{12})}. \quad (6.38)$$

The reader may consult [2][3][4] for detailed properties of this monoid algebra and how it is used for both perturbative and non-perturbative computations in string field theory. In our formulation here this algebra is consistent with the  $\text{OSp}(d|2)$  supersymmetry, and therefore we use supertrace and superdeterminants instead of the trace and determinant in [2][3][4].

The results in (6.36-6.38) were computed for the Moyal star product in a discrete mode space, but the same formal result applies also with our new Moyal star product in  $\sigma$ -space. In the present case the non-commutative variables are labelled by  $i$  which is a combination of discrete ( $M$ ) and continuous ( $\sigma$ ) labels. The shifted gaussian in our new formalism is

$$A_{\mathcal{N},M,\lambda} \equiv \mathcal{N} \exp \left( - \left( \int_0^\pi \int_0^\pi d\sigma d\sigma' \xi^i(\sigma) M_{ij}(\sigma, \sigma') \xi^j(\sigma') \right) - \int_0^\pi d\sigma \xi^i(\sigma) \lambda_i(\sigma) \right) \quad (6.39)$$

where  $\xi^i(\sigma) = (x^M(\sigma), p_M(\sigma))$  are the string half-phase degrees of freedom,  $M_{ij}(\sigma, \sigma')$  is a complex square matrix and  $\lambda_i(\sigma)$  is a complex column matrix. Under the Moyal star product they form a closed algebra called a monoid (which is almost a group, except for the inverse condition). Taking into account the star commutation rules in (3.25) we identify the matrix  $s^{ij}$  with the new type of labels as follows

$$s^{ij} = \begin{matrix} i= \\ j= x^N(\sigma') \quad , \quad p_N(\sigma') \end{matrix} \begin{pmatrix} x^M(\sigma) & 0 & i^{1-N} \delta_N^M \hat{\delta}_{+-}(\sigma, \sigma') \\ p_M(\sigma) & (-i)^{1-N'} \delta_M^N \hat{\delta}_{-+}(\sigma, \sigma') & 0 \end{pmatrix} \quad (6.40)$$

where  $\hat{\delta}_{+-}(\sigma, \sigma')$  is given in Eqs.(3.26-3.27), while  $\hat{\delta}_{-+}(\sigma, \sigma')$  is the transpose of the “matrix”  $\hat{\delta}_{+-}$  and then re-labelled by replacing  $\sigma \leftrightarrow \sigma'$ . The parameters  $M_{ij}$  and  $\lambda_i$  of the shifted gaussian take the form

$$M_{ij} = \begin{matrix} i= \\ j= x^N(\sigma') \quad , \quad p_N(\sigma') \end{matrix} \begin{pmatrix} x^M(\sigma) & a_{MN}(\sigma, \sigma') & b_M^N(\sigma, \sigma') \\ p_M(\sigma) & b_N^M(\sigma, \sigma') & d^{MN}(\sigma, \sigma') \end{pmatrix}, \quad \lambda_i = \begin{pmatrix} \lambda_{x^M}(\sigma) \\ \lambda_{p_M}(\sigma) \end{pmatrix} \quad (6.41)$$

where the diagonal entries of  $M$  are (super)symmetric matrices while the off diagonal entries are related by a (super) transposition

$$b_N^M(\sigma, \sigma') = (-1)^{MN} b_M^N(\sigma', \sigma). \quad (6.42)$$

The matrix  $m = Ms$  becomes

$$m_i^j = M_{ik} s^{kj} = \begin{pmatrix} b_M^N(\sigma, \sigma') \text{sign}(\pi/2 - \sigma') & a_{MN}(\sigma, \sigma') \text{sign}(\pi/2 - \sigma') (-1)^N \\ d^{MN}(\sigma, \sigma') \text{sign}(\pi/2 - \sigma') & b_N^M(\sigma, \sigma') \text{sign}(\pi/2 - \sigma') (-1)^N \end{pmatrix} \quad (6.43)$$

Then they are combined according to the rules (6.36-6.38) to obtain the result for the monoid algebra.

As an example, for perturbative computations, the matrix  $M_{ij}(\sigma, \sigma')$  is fairly simple. It follows from the vacuum state given in (6.13)

$$M_{ij}^{pert} = \begin{pmatrix} g_{MN} \frac{|\partial_\sigma|}{\pi} \delta_\varepsilon^+(\sigma, \sigma') & 0 \\ 0 & g^{MN} \frac{\pi}{|\partial_\sigma|} \delta_\varepsilon^-(\sigma, \sigma') \end{pmatrix} \quad (6.44)$$

In general the *regulated* delta function in (6.44) satisfies D-brane boundary conditions as discussed in section (IV A). The regulator  $\varepsilon$  insures that all computations are well defined.

The regulator is removed after renormalization of the cubic coupling constant  $g_0$  as shown in [5]. For example, for the D25 brane we would use in (6.44) the  $\delta_\varepsilon^{\pm nn}(\sigma, \sigma')$  in all directions  $M$ . But if more complicated D-brane boundary conditions are desired, then in the corresponding directions  $M$  we would use the  $\delta_\varepsilon^{\pm nn}$  or  $\delta_\varepsilon^{\pm dd}$  given in section (IV A). In this way, by making only minimal changes through the regulated delta functions, non-trivial D-brane boundary conditions are implemented easily in our new MSFT formalism.

If the  $M$  in (6.44) is used in monoid computations directly in the form shown in (6.44) with  $\delta^{nn}$ , then this approach reproduces all the results of the computations obtained previously in [2][3][4][5], including the *off-shell* 4-tachyon (Veneziano) scattering amplitude.

## VII. OUTLOOK

The central structure in this paper is the new Moyal  $\star$  product in the  $\sigma$ -basis in Eq.(2.9) that implements the interactions of strings. The string fields  $A(x, p)$  that are multiplied with this product are labeled by *half of the phase space* of the string  $(x_+^M(\sigma), p_{-M}(\sigma))$  as opposed to the full phase space  $(X^M(\sigma), P_M(\sigma))$ . The label  $M = (\mu, b, c)$  includes both the spacetime “matter  $\mu$ ” and the  $(b, c)$  ghosts in an  $\text{OSp}(d|2)$  covariant notation. The star product  $\star$ , which is independent of the details of any conformal field theory on the worldsheet (CFT), is background independent and is invariant under this supersymmetry for all CFTs.

The symmetric  $x_+^M(\sigma) = \frac{1}{2}(X^M(\sigma) + X^M(\pi - \sigma))$  and the antisymmetric  $p_{-M}(\sigma) = \frac{1}{2}(P_M(\sigma) - P_M(\pi - \sigma))$  commute with each other in the quantum mechanics (QM) of the first quantized string, and therefore they are simultaneous observables in QM. The eigenvalues  $(x_+, p_-)$  of these simultaneous observables provide a complete set of labels for the first quantized string states  $\langle x_+^M(\sigma), p_{-M}(\sigma) |$ . The string field  $A(x, p)$  corresponds to the probability amplitude of a general string state  $|A\rangle$  that has the given phase space configuration  $A(x, p) = \langle x_+^M(\sigma), p_{-M}(\sigma) | A \rangle$ . Hence the Moyal product in Eq.(2.9) which creates a non-commutativity in the space of eigenvalues  $(x_+^M(\sigma), p_{-M}(\sigma))$  has nothing to do with the Moyal product in QM despite the close mathematical similarities. However this close similarity is interpreted in this paper as an induced quantum mechanics (iQM) that governs the fundamental *string interactions*.

To be able to compute reliably in string field theory we need a regulator to obtain unambiguous results. The essential role of the regulator is to tame some singularities associated

with the midpoint of the string at  $\pi/2$ . In this paper the regulator is the dimensionless small parameter  $\varepsilon$ . We regulated at first the QM quantum operators of the first quantized string by defining  $\hat{X}(\sigma, \varepsilon)$ , while no regularization is needed for the operator  $\hat{P}_M(\sigma)$ , as discussed in section (IV). Consequently, the eigenvalues of the half phase space are also regularized, so that the string field is regularized with the  $\varepsilon$  through its labels  $A(x_+^M(\sigma, \varepsilon), p_{-M}(\sigma))$ . On this regularized basis we showed how to represent all QM operators that belong to any CFT, by their string-field-counterparts in the induced iQM. This representation involves only star products of the string fields. In particular some crucial operators that are needed to construct string field theory, i.e. the stress tensor  $T_{\pm\pm}$ , the BRST current  $j_B$  and the BRST operator, for any CFT are constructed as *regularized string fields* that operate in the induced iQM. Using them we constructed the regularized action for the new Moyal string field theory (MSFT).

An important aspect of the new formulation is that the regularized midpoint of the string  $x_+^M(\pi/2, \varepsilon)$  is not isolated from the rest of the string degrees of freedom in its treatment under the  $\star$  product. Nevertheless the new product has the magical property that the midpoint  $x_+^M(\pi/2, \varepsilon)$  acts trivially as a complex number (no derivatives induced on the field) when it is star-multiplied with any string field  $x_+^M(\pi/2, \varepsilon) \star A(x, p) = A(x, p) \star x_+^M(\pi/2, \varepsilon) = x_+^M(\pi/2, \varepsilon) A(x, p)$ . Remarkably, this important property of the midpoint in string joining is automatically implemented by the new Moyal  $\star$ .

We constructed an infinite number of solutions to the non-perturbative equation of motion  $\bar{A} \star \bar{A} = 0$  of the purely cubic theory, in the form  $\bar{A}_{sol}(x, p) = Q(x, p)$ , where the *BRST field*  $Q(x, p)$  is derived with our methods from any CFT on the worldsheet. We turned this observation into a method for finding an infinite number of non-perturbative solutions to the string field equation,  $\hat{Q}A + g_0 A \star A = 0$ , in the form,  $g_0 A = Q_2 - Q_1$ , where  $Q_i(x, p)$  correspond to the *BRST fields* (as obtained with our methods) of a pair of conformal field theories  $CFT_i$ .

We have shown that all successful computations previously accomplished in MSFT using the old regularized Moyal star product (which was tied to the flat CFT in  $d = 26$ , and treated the midpoint as special), are also reproduced by the new formalism when the same flat CFT is used. However the advantage of the new approach is that it also applies to any curved CFT and can easily include the effects of D-brane boundary conditions. The applications of the new features will be explored in future work in several directions as follows.

It would be a very interesting exercise to apply our formalism to some simple cases of exact CFTs. Sometime ago some of the earliest examples of exact conformal field theories that describe strings in curved spaces with *one time coordinate* were suggested [23] and studied at the classical and first quantized levels [24]-[26]. The BRST operator  $Q(x, p)$  associated with such models is constructed in terms of a current algebra (or Kac-Moody algebra) basis which replaces the half phase space  $(x(\sigma), p(\sigma))$ . Using the properties of the current algebra should be an important tool to compute in these special curved spaces using our MSFT approach.

We are eager to aim our new MSFT approach to investigate the physical circumstances in which string theory should play its most important physical role. Noting that string field theory (SFT) is a complete approach that incorporates generally both the curved background and the interactions in the non-perturbative description of the theory, it is very important to pursue the SFT avenue despite the fact that computations may be difficult. The areas that we think are important to investigate with our MSFT formalism includes very early cosmology in the vicinity of cosmological singularities as well as black hole or black D-brane type singularities. In particular, there has been some new developments in identifying uniquely cosmological backgrounds that are geodesically complete across cosmological singularities [32] to which we plan to apply our formalism.

An understanding of the very early cosmology of the universe through string theory has been traditionally a hope that it would eventually yield an explanation of why we live in four dimensions and provide the ingredients of the Standard Model of particle physics, such as the number of generations and their symmetry structures. It is believed that string physics is unavoidable in the deeply small and highly curved quantum mechanical regions of space-time. In this paper we have developed sharper tools to address such issues in the context of the new MSFT, including string-string interactions both perturbatively and non-perturbatively, and hope to make further progress in the pursuit of these goals.

We wish to conclude with a speculation on the origin of quantum mechanics. A by-product of our approach is an astonishing suggestion of the formalism: the roots of ordinary quantum mechanics may originate from the non-commutative interactions in string theory. Indeed, the string joining Moyal star induces non-commutativity between otherwise commutative string degrees of freedom  $(\hat{x}_+(\sigma, \varepsilon), \hat{p}_-(\sigma))$ . We draw again the attention of the reader to the remarkable property that even though the operators  $(\hat{x}_+(\sigma, \varepsilon), \hat{p}_-(\sigma))$  commute in



QM, their eigenvalues  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  do not commute with each other under the induced iQM as seen in Eq.(3.25). This non-commutativity under the string-joining star product is what led to the representation of the QM operators in the basis of iQM as in Eqs.(3.36,3.37) in the language of only the string joining star product. The reader is invited to read section (III A) in reverse by starting from Eqs.(3.36,3.37) and interpreting its contents as the emergence of QM from iQM rather than the other way around. Then it seems astonishing that the half-phase-space  $(x_+(\sigma, \varepsilon), p_-(\sigma))$  in iQM under the string joining  $\star$  generates the conventional QM commutation rules for the full phase space operators  $(\hat{X}(\sigma, \varepsilon), \hat{P}(\sigma))$ . Assuming that string theory is right that it underlies all physics, it is then very tempting to speculate that the source of QM rules in all physics may simply be the rules of interactions in string theory as seen explicitly in our paper. This could be the long sought explanation of where QM comes from. This exciting point is very important in its own right, and it will be pursued further.

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## VIII. APPENDIX

### A. Old Discrete Basis Versus New $\sigma$ -Basis

In this section we are going to show that the new  $\sigma$ -basis formalism that does not distinguish the midpoint is equivalent to the old discrete basis that distinguished the midpoint. The old formalism was developed in [2][3][4][5]. By not distinguishing the midpoint we have a cleaner and more efficient approach in practical computations.

The unregulated position and momentum degrees of freedom  $x_+(\sigma)$  and  $p_-(\sigma)$  in the new  $\sigma$  basis can be written in terms of modes with even and odd *cosine* expansions respectively,

$$x_+(\sigma) = x_0 + \sqrt{2} \sum_e x_e \cos e\sigma; \quad p_-(\sigma) = \frac{\sqrt{2}}{\pi} \sum_o p_o \cos o\sigma, \quad (8.1)$$

where  $e = 2, 4, 6, \dots$  and  $o = 1, 3, 5, \dots$ . Their Moyal  $\star$  commutator, using the new  $\star$

product (2.9), is the delta-function that has two forms (3.25)

$$[x_+(\sigma), p_-(\sigma')]_{\star_{new}} = i\delta^{+nn}(\sigma, \sigma') \varepsilon(\pi/2 - \sigma') = i\varepsilon(\pi/2 - \sigma) \delta^{-nn}(\sigma, \sigma'). \quad (8.2)$$

Using the mode expansion for these delta functions we can compare the two results

$$\left( \frac{2}{\pi} + \frac{4}{\pi} \sum_e \cos e\sigma \cos e\sigma' \right) \varepsilon(\pi/2 - \sigma') = \varepsilon(\pi/2 - \sigma) \frac{4}{\pi} \sum_o \cos o\sigma \cos o\sigma'. \quad (8.3)$$

It can be verified that this is an identity [3]. The commutator contains two terms, one with and one without the zero mode:

$$[x_+(\sigma), p_-(\sigma')]_{\star_{new}} = \frac{\sqrt{2}}{\pi} \sum_o [x_0, p_o]_{\star_{new}} \cos o\sigma' + \frac{2}{\pi} \sum_{e,o} [x_e, p_o]_{\star_{new}} \cos e\sigma \cos o\sigma', \quad (8.4)$$

By comparing the Fourier modes to the answer (8.2) we extract the mode commutators and find that they are expressed in terms of the matrix elements of the special matrix  $T$  introduced in [2]

$$[x_0, p_o]_{\star_{new}} = 2iT_{0o}, \quad [x_e, p_o]_{\star_{new}} = 2iT_{eo}, \quad (8.5)$$

where

$$T_{eo} = \frac{4}{\pi} \int_0^{\pi/2} d\sigma \cos e\sigma \cos o\sigma = \frac{4}{\pi} \frac{o(-1)^{(e-o-1)/2}}{e^2 - o^2}, \quad (8.6)$$

$$T_{0o} = \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} d\sigma \cos o\sigma = \frac{2\sqrt{2}}{\pi o} \sin \frac{o\pi}{2} = \frac{2\sqrt{2}}{\pi} \frac{(-1)^{(o-1)/2}}{o}. \quad (8.7)$$

A useful identity is [3] (a sum over  $e$  is implied)

$$\cos o\sigma = \varepsilon\left(\frac{\pi}{2} - \sigma\right) \left( \cos e\sigma - \cos e\frac{\pi}{2} \right) T_{eo}. \quad (8.8)$$

These star commutators for the modes (8.5-8.7) that were obtained with the new Moyal product (2.9) in the  $\sigma$  basis are identical to those given by the old Moyal product in the discrete basis *for the flat CFT case*, once we write  $p_o = p_e T_{eo}$  where  $p_e$  is introduced [2] as a convenient change of basis (it does *not* mean the even momentum modes in  $p_+(\sigma)$ ). Hence the old and new Moyal products are equivalent for the flat CFT. But the new  $\sigma$  basis  $\star$  product (2.9) is more powerful since it is valid for all CFTs due to the fact that the phase space notation in the  $\sigma$  basis is independent of the background fields.

There is one more subtle point in this equivalence which is related to the midpoint. In the old approach the midpoint,

$$\bar{x} \equiv x(\pi/2) = x_0 + \sqrt{2} \sum_{e \geq 2} x_e \cos \frac{e\pi}{2}, \quad (8.9)$$

was distinguished as an independent mode (instead of  $x_0$ ) and did not appear in the old star product which was expressed only in terms of  $x_e, p_e$  (where  $p_e$  is related to  $p_o = p_e T_{eo}$ ). Then the midpoint  $\bar{x}$  had a trivial star product (i.e. it acted on fields with the ordinary product for complex numbers) and in particular commuted with the momentum modes under the string joining  $\star$ . We want to verify that the new  $\star$  product also has the same properties for the midpoint even though the new  $\star$  (2.9) does not distinguish the midpoint. It is useful to note the following notation for  $w_e, v_o$  and the relations among these symbols and the midpoint

$$x_0 = \bar{x} + w_e x_e, \text{ with } w_e = -\sqrt{2} \cos \frac{e\pi}{2} = -\sqrt{2} (-1)^{e/2}, \quad (8.10)$$

$$v_o \equiv T_{0o} = w_e T_{eo}, \quad (8.11)$$

where a summation over  $e$  is implied. The last equation follows from integrating both sides of (8.8). Note that  $v_o$  or equivalently  $T_{0o}$  is really related to the midpoint at  $\pi/2$  since

$$[\bar{x}, p_o]_{\star_{new}} = [(x_0 - w_e x_e), p_o]_{\star} = 2i (T_{0o} - w_e T_{eo}) = 0. \quad (8.12)$$

This verifies the last crucial point in the equivalence of the old and new star products: even though the midpoint  $\bar{x}$  was not distinguished in the new star product,  $\bar{x}$  behaves as if it is insensitive to the derivatives implicit in the new Moyal product and therefore it acts just like a complex number under the new  $\star$ , which is the same as under the old  $\star$ .

It is also instructive to verify the equivalence in reverse. That is, suppose we are given the commutators (8.5-8.7) and (8.12) under the old product  $\star_{old}$  with the same result, and let us derive the local commutators in the continuous  $\sigma$ -basis (8.2) by using only the old star product. By writing out  $x_+(\sigma), p_-(\sigma')$  in terms of modes (8.1) and using (8.5-8.7) with  $\star_{old}$  we get the same form as (8.4)

$$[x_+(\sigma), p_-(\sigma')]_{\star_{old}} = 2i \frac{\sqrt{2}}{\pi} \sum_o T_{0o} \cos o\sigma' + 2i \frac{2}{\pi} \sum_{e,o} T_{eo} \cos e\sigma \cos o\sigma'. \quad (8.13)$$

Now insert the results (8.6-8.8) for  $T_{0o}, T_{eo}$  given in the old literature [2] and compute the

right hand side of this equation. We find

$$\begin{aligned}
&= \left[ \begin{aligned} &2i \frac{\sqrt{2}}{\pi} \sum_o \left( \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} d\sigma_1 \cos o\sigma_1 \right) \cos o\sigma' \\ &+ 2i \frac{2}{\pi} \sum_{e,o} \left( \frac{4}{\pi} \int_0^{\pi/2} d\sigma_1 \cos e\sigma_1 \cos o\sigma_1 \right) \cos e\sigma \cos o\sigma' \end{aligned} \right] \\
&= i \frac{2}{\pi} \int_0^{\pi/2} d\sigma_1 \delta^-(\sigma_1, \sigma') + i \int_0^{\pi/2} d\sigma_1 \left( \delta^+(\sigma_1, \sigma) - \frac{2}{\pi} \right) \delta^-(\sigma_1, \sigma') \\
&= i \delta^{+nn}(\sigma, \sigma') \varepsilon(\pi/2 - \sigma'). \tag{8.14}
\end{aligned}$$

Therefore, the old product  $\star_{old}$  reproduces the new product  $\star_{new}$  in the  $\sigma$  basis.

Hence the same string field can be rewritten in the different bases

$$A(x_+(\sigma), p_-(\sigma)) = A(x_0, x_e, p_o) = A(x_0, x_e, p_o)|_{x_0=\bar{x}+w_e x_e} = \tilde{A}(\bar{x}, x_e, p_o). \tag{8.15}$$

and the joining of strings can be expressed equivalently in either the old or the new star products if the CFT is flat. To show some subtleties of how this works, we begin with the relation between the old and new bases. The old star product is (using (2.2) with fixed  $\bar{x}$ )

$$\tilde{A}_1(\bar{x}, x_e, p_o) \star_{old} \tilde{A}_2(\bar{x}, x_e, p_o) = \tilde{A}_1(\bar{x}, (x'_e + iT_{eo}\partial_{p_o}), (p'_o - iT_{eo}\partial_{x_e})) \star \tilde{A}_2(\bar{x}, x_e, p_o) \tag{8.16}$$

where in the last formula we used Eq.(2.2). Rewrite this in terms of the new basis  $A_{1,2}(x_0, x_e, p_o)$  making explicit that  $x_0$  is a function of  $\bar{x}$  and  $x_e$  when the old star product is used:

$$\tilde{A}_1(\bar{x}, x_e, p_o) \star_{old} \tilde{A}_2(\bar{x}, x_e, p_o) \tag{8.17}$$

$$= A_1(x_0(\bar{x}, x_e), x_e, p_o) \star_{old} A_2(x_0(\bar{x}, x_e), x_e, p_o) \tag{8.18}$$

$$= A_1((\bar{x}' + w_e(x'_e + iT_{eo}\partial_{p_o})), (x'_e + iT_{eo}\partial_{p_o}), (p'_o - iT_{eo}\partial_{x_e})) A_2(x_0(\bar{x}, x_e), x_e, p_o) \tag{8.19}$$

$$= A_1\left(\left(x'_0 + \frac{i}{2}\partial_{\bar{p}}\right), (x'_e + iT_{eo}\partial_{p_o}), (p'_o - iT_{eo}(\partial_{x_e} + w_e\partial_{x_0}))\right) A_2(x_0, x_e, p_o). \tag{8.20}$$

$$= A_1(x_0, x_e, p_o) \star_{new} A_2(x_0, x_e, p_o) \tag{8.21}$$

where the last step is proven below. The following manipulations are used in obtaining (8.20) from the previous line: for the first factor,  $(x'_0 + \frac{i}{2}\partial_{\bar{p}})$ , note

$$\begin{aligned}
\bar{x}' + w_e(x'_e + iT_{eo}\partial_{p_o}) &= x'_0 + iw_e T_{eo}\partial_{p_o} = x'_0 + iT_{0o}\partial_{p_o} = x'_0 + \frac{i}{2}\partial_{\bar{p}}, \\
\text{with } \partial_{\bar{p}} &\equiv 2T_{0o}\partial_{p_o} = 2v_o\partial_{p_o} = \frac{1}{\pi} \int_0^\pi d\sigma \varepsilon\left(\frac{\pi}{2} - \sigma\right) \partial_{p_-(\sigma)},
\end{aligned} \tag{8.22}$$

and for the last factor note,

$$(p'_o - iT_{eo}\partial_{x_e}) A_2(x_0(\bar{x}, x_e), x_e, p_o) = \left(p'_o - iT_{eo} \left(\frac{\partial x^0}{\partial x_e} \partial_{x_0} + \partial_{x_e}\right)\right) A_2, \tag{8.23}$$

where the  $\partial_{x_0}$  term in (8.23) takes care of applying the derivative  $T_{eo}\partial_{x_e}$  on the  $x_e$  in  $x_0(\bar{x}, x_e)$  by using the chain rule, with

$$\frac{\partial x^0}{\partial x_e} = w_e, \text{ and } \partial_{x_0} = \frac{1}{2} \int_0^\pi d\sigma \partial_{x_+(\sigma)}, \quad (8.24)$$

while the last  $\partial_{x_e}$  in (8.23) is applied to  $x_e$  which is not inside  $x_0(\bar{x}, x_e)$  in  $A_2$ . In this way we have established in general the equivalence of the old and new star products (8.17) and (8.21). Any computation can be performed by switching between the old/new versions of the star as long as sufficient care is used as demonstrated above.

On the way to prove the equivalence of (8.20) and the new  $\star$  product in the  $\sigma$ -basis via (8.21), we first note that the expression in (8.20) is reproduced from (8.21) by defining a new star product in the  $(x_0, x_e, p_o)$  mode basis that includes the center of mass mode  $x_0$  that is explicitly shown as follows

$$\star_{new} = (\star_{x_e, p_o}) \exp \left( \frac{i}{2} \left( \overleftarrow{\partial}_{x_0^M} \overrightarrow{\partial}_{\bar{p}_M} - \overleftarrow{\partial}_{\bar{p}_M} \overrightarrow{\partial}_{x_0^M} \right) \right), \quad (8.25)$$

where  $\partial_{\bar{p}_M} = 2v_o \partial_{p_{Mo}}$  is defined as in (8.22), and  $(\star_{x_e, p_o})$  is the non-zero mode contribution given by

$$(\star_{x_e, p_o}) = \exp \left( iT_{eo} \left( \overleftarrow{\partial}_{x_e^M} \overrightarrow{\partial}_{p_{Mo}} - \overleftarrow{\partial}_{p_{Mo}} \overrightarrow{\partial}_{x_e^M} \right) \right). \quad (8.26)$$

In particular, as applications of this mode-version of the new star, note that for products with the center of mass mode,  $x_0^\mu$  or  $x_0^b$ , we get

$$x_0^M \star_{new} A(x_0, x_e, p_o) = \left( x_0^M + \frac{i}{2} \partial_{\bar{p}_M} \right) A(x_0, x_e, p_o), \quad (8.27)$$

however, for products with the midpoint we get (as in (8.12))

$$\bar{x}^M \star_{new} A(x_0, x_e, p_o) = \bar{x}^M A(x_0, x_e, p_o), \quad (8.28)$$

where there are no derivatives on the right hand side, showing again that the midpoint  $\bar{x}$  is insensitive to the derivatives in the new star product, and acts just like a complex number.

Finally, to prove the equality between the star products in the new  $\sigma$ -basis and the new mode basis (8.25) we use the mode expansions of the position and momentum (8.1) and

compute the mode expansion of their derivatives by using the chain rule as follows:

$$\begin{aligned} x_+(\sigma) &= x_0 + \sqrt{2} \sum_e x_e \cos e\sigma, \\ \partial_{x_+(\sigma)} &= \frac{2}{\pi} (\partial_{x_0} + \sqrt{2} \sum_e \cos e\sigma \partial_{x_e}) \end{aligned} \quad (8.29)$$

$$\begin{aligned} \partial_{x_+(\sigma)} x_+(\sigma') &= \delta^{+nn}(\sigma, \sigma'), \\ p_-(\sigma) &= \frac{\sqrt{2}}{\pi} \sum_o p_o \cos o\sigma, \\ \partial_{p_-(\sigma)} &= 2\sqrt{2} \sum_o \cos o\sigma \partial_{p_o}, \\ \partial_{p_-(\sigma)} p_-(\sigma') &= \delta^{-nn}(\sigma, \sigma'). \end{aligned} \quad (8.30)$$

The first term (and similarly the second term) in the exponential of the new star product in (2.9) can be evaluated in terms of modes

$$\begin{aligned} & \frac{i}{4} \int_0^\pi d\sigma \vec{\partial}_{p_-(\sigma)} \cdot \overleftarrow{\partial}_{x(\sigma)} \varepsilon \left( \frac{\pi}{2} - \sigma \right) \\ &= \frac{i}{4} \int_0^\pi d\sigma \varepsilon \left( \frac{\pi}{2} - \sigma \right) \left( 2\sqrt{2} \sum_o \cos o\sigma \vec{\partial}_{p_o} \right) \cdot \left( \frac{2}{\pi} \left( \overleftarrow{\partial}_{x_0} + \sqrt{2} \sum_e \cos e\sigma \overleftarrow{\partial}_{x_e} \right) \right) \\ &= \left( \frac{i}{4} \frac{2}{\pi} 2\sqrt{2} \int_0^\pi d\sigma \varepsilon (\pi/2 - \sigma) \cos o\sigma \right) \vec{\partial}_{p_o} \cdot \overleftarrow{\partial}_{x_0} + \text{''} \vec{\partial}_{p_o} \overleftarrow{\partial}_{x_e} \text{''} \\ &= \left( \frac{i}{4} \frac{2}{\pi} 2\sqrt{2} \int_0^{\pi/2} d\sigma \cos o\sigma \right) \vec{\partial}_{p_o} \cdot \overleftarrow{\partial}_{x_0} + \text{''} \vec{\partial}_{p_o} \overleftarrow{\partial}_{x_e} \text{''} \\ &= iT_{0o} \vec{\partial}_{p_o} \cdot \overleftarrow{\partial}_{x_0} + iT_{eo} \vec{\partial}_{p_o} \cdot \overleftarrow{\partial}_{x_e}. \end{aligned} \quad (8.31)$$

where we used  $\text{''} \vec{\partial}_{p_o} \overleftarrow{\partial}_{x_e} \text{''}$  as a short notation to include the integrals that are not shown explicitly to save space. Putting together all the terms in the exponential of (2.9), leads exactly to Eq. (8.25), thus proving that the  $\star$  in the continuous  $\sigma$ -basis (2.9) and the  $\star$  in the discrete basis (8.25) that includes the center of mass mode are identical.

Therefore, we have proven (noting the relation (8.15) between  $A$  and  $\tilde{A}$ ) that

$$\begin{aligned} & \tilde{A}_1(\bar{x}, x_e, p_o) \star_{old} \tilde{A}_2(\bar{x}, x_e, p_o), \\ &= A_1(x_0, x_e, p_o) \star_{new} A_2(x_0, x_e, p_o), \\ &= A_1(x_+(\sigma), p_-(\sigma)) \star_{new} A_2(x_+(\sigma), p_-(\sigma)). \end{aligned} \quad (8.32)$$

In the  $\star_{new}$  the center of mass mode  $x_0$  is active in the string joining as an independent degree of freedom; this is in contrast to  $\star_{old}$  where the midpoint  $\bar{x}$  is passive as an independent degree of freedom. We have shown that  $\star_{old}$  and  $\star_{new}$  are completely equivalent, but they must be used consistently in their own basis. As we saw above in Eq.(8.20),  $\partial_{x_e}$  does not mean the same thing in the various bases because partial derivatives imply that some variables are fixed while evaluating derivatives, but the quantities held fixed are different in the various bases:  $\bar{x}$  fixed in the old basis, while  $x_0$  fixed in the new basis. Hence one must be careful in

such computations. Clearly, as long as we stick consistently to only one basis, or be careful in the translation as in the steps in Eqs.(8.17-8.21), there will be no errors.

The advantage of the new star is that there is a lot of simplification in the  $\sigma$  formalism because the midpoint does not need to be distinguished. Furthermore, the  $\sigma$  basis is clearly background independent and applies to all CFT backgrounds that may be used in the constructions of the BRST field  $Q(x, p)$ . Moreover, quantum operators products that are well known in the QM of any CFT apply directly also in the parallel induced iQM of the MSFT formalism, thus rendering the computations in the new MSFT much easier.

## B. The BRST Gauge Transformations for an Invariant Action

The MSFT action (5.56) has the following form when the midpoint integration is made explicit

$$\begin{aligned} S &= -Str' \int d\bar{x}^b \left( A \star Q \star A + \frac{g_0}{3} A \star \partial_{\bar{x}^b} A \star \partial_{\bar{x}^b} A \right) \\ &= -Str' \left( \partial_{\bar{x}^b} (A \star Q \star A) + \frac{g_0}{3} \partial_{\bar{x}^b} A \star \partial_{\bar{x}^b} A \star \partial_{\bar{x}^b} A \right), \end{aligned} \quad (8.33)$$

$Str'$  is the remainder of the phase space integration that does not include the midpoint ghost modes. The gauge transformation that leaves this action invariant is the following

$$\delta_\Lambda A = [Q, \Lambda]_\star + g_0 \{ \partial_{\bar{x}^b} A, \partial_{\bar{x}^b} \Lambda \}_\star. \quad (8.34)$$

Analyzing the ghost numbers in Eq. (8.34) we conclude that the gauge parameter  $\Lambda$  is bosonic with ghost number  $-2$ .

Let us check the invariance under the transformation (8.34).

$$\delta_\Lambda S = -Str' \partial_{\bar{x}^b} \left[ \begin{aligned} &(\delta_\Lambda A \star Q \star A + A \star Q \star \delta_\Lambda A) \\ &+ g_0 \delta_\Lambda A \star (\partial_{\bar{x}^b} A)_\star^2 \end{aligned} \right] \quad (8.35)$$

$$\begin{aligned} &= -Str' \partial_{\bar{x}^b} \left[ \begin{aligned} &\left( (Q \star \Lambda - \Lambda \star Q + g_0 \{ \partial_{\bar{x}^b} A, \partial_{\bar{x}^b} \Lambda \}_\star) \star Q \star A \right) \\ &+ A \star Q \star (Q \star \Lambda - \Lambda \star Q + g_0 \{ \partial_{\bar{x}^b} A, \partial_{\bar{x}^b} \Lambda \}_\star) \end{aligned} \right] \\ &\quad \left[ + g_0 (Q \star \Lambda - \Lambda \star Q + g_0 \{ \partial_{\bar{x}^b} A, \partial_{\bar{x}^b} \Lambda \}_\star) \star (\partial_{\bar{x}^b} A)_\star^2 \right] \end{aligned} \quad (8.36)$$

Next we use the cyclic property of the supertrace, and the star nilpotency of the BRST field

$Q \star Q = 0$ , to cancel kinetic terms and reorganize the interaction terms

$$\delta_\Lambda S = Str' \left[ \begin{aligned} & \partial_{\bar{x}^b} \left( (Q \star \Lambda \star Q \star A - Q \star \Lambda \star Q \star A) \right. \\ & \quad \left. + (-\Lambda \star Q_\star^2 \star A + A \star Q_\star^2 \star \Lambda) \right) \\ & + g_0 \partial_{\bar{x}^b} \left( Q \star \begin{bmatrix} -\partial_{\bar{x}^b} (A \star \partial_{\bar{x}^b} A \star \Lambda) \\ +\partial_{\bar{x}^b} (A \star \partial_{\bar{x}^b} \Lambda \star A) \\ +\partial_{\bar{x}^b} (\Lambda \star \partial_{\bar{x}^b} A \star A) \end{bmatrix} \right) \end{aligned} \right] \quad (8.37)$$

The first term vanishes explicitly while the second term takes the form

$$\delta_\Lambda S = g_0 Str' [\partial_{\bar{x}^b} Q \star \partial_{\bar{x}^b} (\Lambda \star \partial_{\bar{x}^b} A \star A + A \star \partial_{\bar{x}^b} \Lambda \star A - A \star \partial_{\bar{x}^b} A \star \Lambda)]. \quad (8.38)$$

In the next step we expand the string field  $A$  and the gauge parameter  $\Lambda$  in the powers of  $\bar{x}^b$ :

$$A = \bar{x}^b A^{(0)} + A^{(-1)}, \quad \Lambda = \bar{x}^b \Lambda^{(-1)} + \Lambda^{(-2)}. \quad (8.39)$$

The BRST charge contributes

$$\partial_{\bar{x}^b} Q \equiv Q_{++} = \int_0^{\pi/2} d\sigma p_{-b}(\sigma) \star x_+^c(\sigma). \quad (8.40)$$

where  $Q_{++}$  has ghost number +2. Therefore,  $\delta_\Lambda S$  can be reorganized into the form

$$\delta_\Lambda S = g_0 Str' \{Q_{++} \Psi\},$$

with

$$\Psi = [\Lambda^{(-2)}, (A^{(0)})^2]_\star + \{A^{(-1)}, \{\Lambda^{(-1)}, A^{(0)}\}\}_\star. \quad (8.41)$$

We dropped the star product in  $Str'(Q_{++} \star \Psi)$  because it is allowed under the supertrace.

Next we examine this  $Str'$ . Making the Grassmann integrals explicit as being equivalent to derivatives, we can write

$$\begin{aligned} Str'(Q_{++} \Psi) &= \\ &= Tr \left\{ \prod_{\sigma'} \left[ \frac{\partial}{\partial p_{-b}(\sigma')} \frac{\partial}{\partial x_+^b(\sigma')} \frac{\partial}{\partial p_{-c}(\sigma')} \frac{\partial}{\partial x_+^c(\sigma')} \left( \int_0^{\pi/2} d\sigma p_{-b}(\sigma) x_+^c(\sigma) \right) \Psi(p_{-b}, x_+^c, p_{-c}, x_+^b) \right] \right\} \\ &\sim Tr \left( \int_0^{\pi/2} d\sigma \left[ \frac{\partial}{\partial p_{-c}(\sigma)} \frac{\partial}{\partial x_+^b(\sigma)} \Psi(p_{-b}, x_+^c, p_{-c}, x_+^b) \right]_{p_{-b}=x_+^c, p_{-c}=x_+^b=0} \right) \end{aligned}$$

where  $Tr$  is the remaining bosonic integrals. Now, substituting  $\Psi$  from (8.41) the two derivatives in the last line produce several (anti)commutators. To see this we write out the bose/fermi components of  $A$  and  $\Lambda$

$$A = \bar{x}^b A^{(0)} + A^{(-1)}, \quad \Lambda = \bar{x}^b \Lambda^{(-1)} + \Lambda^{(-2)}. \quad (8.42)$$



and then find that  $Str'(Q_{++}\Psi)$  takes the form

$$Tr \int_0^{\pi/2} d\sigma \left( \begin{aligned} & \left[ \frac{\partial}{\partial p_{-c}(\sigma)} \frac{\partial}{\partial x_+^b(\sigma)} \Lambda^{(-2)}, (A^{(0)})^2 \right]_{\star} \\ & + \left[ \Lambda^{(-2)}, \frac{\partial}{\partial p_{-c}(\sigma)} \frac{\partial}{\partial x_+^b(\sigma)} (A^{(0)})^2 \right]_{\star} \\ & - \left\{ \frac{\partial}{\partial x_+^b(\sigma)} \Lambda^{(-2)}, \frac{\partial}{\partial p_{-c}(\sigma)} (A^{(0)})^2 \right\}_{\star} \\ & + \left\{ \frac{\partial}{\partial p_{-c}(\sigma)} \Lambda^{(-2)}, \frac{\partial}{\partial x_+^b(\sigma)} (A^{(0)})^2 \right\}_{\star} \\ & - \left\{ \frac{\partial}{\partial p_{-c}(\sigma)} \frac{\partial}{\partial x_+^b(\sigma)} A^{(-1)}, \{ \Lambda^{(-1)}, A^{(0)} \} \right\}_{\star} \\ & - \left\{ A^{(-1)}, \frac{\partial}{\partial p_{-c}(\sigma)} \frac{\partial}{\partial x_+^b(\sigma)} \{ \Lambda^{(-1)}, A^{(0)} \} \right\}_{\star} \\ & - \left[ \frac{\partial}{\partial x_+^b(\sigma)} \{ \Lambda^{(-1)}, A^{(0)} \}_{\star}, \frac{\partial}{\partial p_{-c}(\sigma)} A^{(-1)} \right]_{\star} \\ & - \left[ \frac{\partial}{\partial x_+^b(\sigma)} A^{(-1)}, \frac{\partial}{\partial p_{-c}(\sigma)} \{ \Lambda^{(-1)}, A^{(0)} \}_{\star} \right]_{\star} \end{aligned} \right) \quad (8.43)$$

It is understood that we should set  $p_{-b} = x_+^c = p_{-c} = x_+^b = 0$  after taking the derivatives. All these terms vanish under the trace on the basis of their bose/fermi properties,  $Tr[a, b] = 0$  when they are both bosons and  $Tr\{\alpha, \beta\} = 0$  when they are both fermions.

Hence, we proved that the gauge transformation that gives  $\delta_\Lambda S = 0$  has the following form

$$\delta_\Lambda A = [Q, \Lambda]_{\star} + g_0 \{ \partial_{\bar{x}^b} A, \partial_{\bar{x}^b} \Lambda \}_{\star}, \quad (8.44)$$

for  $\Lambda(x, p)$  a general field of ghost number  $-2$ , and  $A$  a general field of ghost number  $-1$ .

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